

# ROBUST DISCRETE COMPLEX ANALYSIS: A TOOLBOX

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**ABSTRACT.** We prove a number of uniform double-sided estimates relating discrete counterparts of several classical conformal invariants of a quadrilateral: cross-ratios, extremal lengths and random walk partition functions. These estimates hold true for any simply connected discrete domain (subset of a “roughly regular” planar graph, e.g., the standard square grid) with four marked boundary vertices, and are completely independent of the domain geometry which can be very rough, having many fiords and bottlenecks of various widths. This allows one to use classical methods of geometric complex analysis for discrete domains “staying on the microscopic level”. Applications include a discrete version of the classical Ahlfors-Beurling-Carleman estimate and some “surgery technique” developed for discrete quadrilaterals.

## 1. INTRODUCTION

**1.1. Motivation.** This paper was originally motivated by the recent activity devoted to the analysis of interfaces arising in the critical 2D lattice models (e.g., see [Smi06, Smi10] and references therein), particularly the random cluster representation of the Ising model [Kem09, DHN11, KS12, CDH12, CDHKS12]). At the same time, it has an independent interest, being devoted to one of the central objects of discrete potential theory on a (weighted) graph  $\Gamma$  embedded into a complex plane: partition functions of the underlying random walk running in a discrete (simply connected) domain  $\Omega$ .

Dealing with some 2D lattice model and its scaling limit (an archetypical example is the Brownian motion in  $\Omega$ , which can be realized as the limit of simple random walks on refining square grids  $\delta\mathbb{Z}^2$ ), one usually works in the context when the lattice mesh  $\delta$  tends to zero. Then, it can be argued that a discrete lattice model is sufficiently close to the continuous one, if  $\delta$  is small enough: e.g., random walks hitting probabilities (discrete harmonic measures) converge to those of the Brownian motion (continuous harmonic measure, cf. [Kak44]) as  $\delta \rightarrow 0$ . After rescaling the underlying grid by  $\delta^{-1}$ , statements of that sort provide an information about properties of the random walk running in *large* discrete domains  $\Omega \subset \mathbb{Z}^2$ .

Unfortunately, this setup is not sufficient when we are interested in fine geometric properties of 2D lattice models (e.g., full collection of interfaces in the random cluster

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representation of the critical Ising model): sometimes it turns out that one needs to consider not only macroscopic  $\Omega$ 's but also their subdomains “on all scales (like  $\delta^\varepsilon$  or even several lattice steps) simultaneously” in order to gain some macroscopic information. Questions of that kind are still tractable by classical means if those microscopic parts of  $\Omega$  are regular enough (e.g., rectangular-type subsets of  $\mathbb{Z}^2$ , cf. [KS12]). Nevertheless, if no such regularity assumptions can be made using some monotonicity features of the particular model, the situation immediately becomes much more annoying.

Having in mind the classical geometric complex analysis as a guideline, in our paper we construct its discrete version “staying on a microscopic level” (i.e., without any passage to the limit or coupling arguments), which allows one to handle (possibly, extremely rough) discrete domains by more-or-less the same methods as continuous ones. We prove a number of *uniform estimates* (a “toolbox”) which hold true for any simply connected  $\Omega$ , possibly having many fiords and bottlenecks of various widths, including very thin (several lattice steps) ones. Being interested in estimates rather than convergence, we do not need any nice “complex structure” on the underlying weighted planar graph. Instead, we assume that  $\Gamma$  and the corresponding random walk on it satisfy several rough “elliptic type” estimates (see Sect. 2.1 and 2.3 for details), deriving all results from those assumptions.

It seems worthwhile to consider such “roughly regular” weighted graphs  $\Gamma$ , since we hope that our results can be applied in various setups (e.g., coming from lattice models defined on nonregular lattices, circle packing constructions, or DLA-type processes in nonconstant environments) where some connection to discrete harmonic measure can be established (or is already plugged into the model). It should be said that we essentially use “uniformly bounded degree” and “quantitative local finiteness” assumptions (as well as uniform boundedness of edge weights), especially when proving a duality estimate for discrete extremal lengths. Thus, some important setups (notably, circle packings of random planar maps) are not covered, requiring some additional input (possibly, a kind of a “surgery” near high degree vertices, cf. [GN12]). On the other hand, results of our paper seems to be new even if  $\Gamma = \mathbb{Z}^2$ , so the reader not interested in full generality may always think about this, probably simplest possible, case.

In order to shorten the presentation, below we widely use the following notation: assuming that all “structural parameters” of the graph  $\Gamma$  listed in Sect. 2.1 and 2.3 are fixed once forever (or if we work with some concrete planar graph, e.g.,  $\Gamma = \mathbb{Z}^2$ ),

- by “**const**” we denote *positive* constants (like  $\frac{1}{2\pi}$  or  $30^{383}$ ) which do not depend on geometric properties (the shape of  $\Omega$ , positions of boundary points, etc) of the configuration under consideration or additional parameters we deal with (thus, “ $f \leq \mathbf{const}$ ” means that there exists a positive constant  $C$  such that the inequality  $f \leq C$  holds true *uniformly* over all possible configurations);
- we write “ $f \asymp g$ ” if there exist two positive constants  $C_{1,2}$  such that one has  $C_1 f \leq g \leq C_2 f$  uniformly over all possible configurations (in other words,  $f$  and  $g$  are comparable up to some absolute constants which we do not specify);
- we write, e.g., “if  $f \geq \mathbf{const}$ , then  $g_1 \asymp g_2$ ” iff, for *any* given constant  $c > 0$ , the estimate  $f \geq c$  implies  $C_{1,2} g_1 \leq g_2 \leq C_{2,2} g_1$ , where  $C_{1,2} = C_{1,2}(c) > 0$  may depend on  $c$  but are independent of all other parameters involved.

**1.2. Main results.** The main objects of interest are (discrete) quadrilaterals, i.e., simply connected domains  $\Omega$  with four marked boundary points  $a, b, c, d$  listed counterclockwise. Focusing on quadrilaterals, we are motivated by two reasons. First, in the classical theory this is a “minimal” configuration which has a nontrivial conformal invariant (e.g., all simply connected  $\Omega$ ’s with three marked boundary points are conformally equivalent due to the Riemann mapping theorem). Second, those are archetypical configuration for the 2D lattice models theory, where one often needs to estimate the probability of some crossing-type event between the opposite sides of  $\Omega$ .

Note that, even if  $\Gamma = \mathbb{Z}^2$ , there is a crucial difference between discrete and continuous theories. The latter one is essentially based on conformal mappings and conformal invariance of various quantities, notably the conformal invariance of extremal lengths (see [Ahl73, Chapter 4] and [GM05, Chapter IV]). Using conformal invariance, one typically may rewrite the question originally formulated in  $\Omega$  as the same question for some canonical domain (unit circle, half-plane, rectangle etc.), thus simplifying the problem drastically (e.g., see [GM05, Theorem IV.5.2]). In particular, up to conformal equivalence,  $(\Omega; a, b, c, d)$  can be described by a single real parameter (modulus). Therefore, all conformal invariants of those  $\Omega$ ’s (cross-ratios, extremal lengths, partition functions of the Brownian motion) are just some concrete functions of each other.

This picture changes completely when coming down to the discrete level: for discrete domains (subsets of a *fixed* graph) we do not have any reasonable notion of conformal equivalence. Nevertheless, for a discrete quadrilateral, one can easily introduce natural analogues of all classical conformal invariants listed above. Namely, let  $\mathbf{Z}_\Omega = Z_\Omega([ab]_\Omega; [cd]_\Omega)$  denotes the total *partition function of random walks* running from the boundary arc  $[ab]_\Omega$  to the opposite arc  $[cd]_\Omega$  inside of  $\Omega$  (if  $\Gamma = \mathbb{Z}^2$ , then  $Z_\Omega$  is the sum of weights  $4^{-\text{Length}(\gamma)}$  of all those nearest-neighbor paths  $\gamma$ , see Sect. 2.4). Then, we define the *discrete cross-ratio*  $\mathbf{Y}_\Omega = Y_\Omega(a, b; c, d)$  of boundary points  $a, b, c, d$  as

$$Y_\Omega := [Z_\Omega(a; d)Z_\Omega(b; c) / Z_\Omega(a; b)Z_\Omega(c; d)]^{\frac{1}{2}}$$

(see Sect. 4.1 for details). We also use the classical definition of *discrete extremal length* (or, equivalently, effective resistance of the corresponding electrical network, see Sect. 6)  $\mathbf{L}_\Omega = L_\Omega([ab]_\Omega; [cd]_\Omega)$  between  $[ab]_\Omega$  and  $[cd]_\Omega$  which goes back to Duffin [Duf62].

Certainly, one cannot hope that  $Z_\Omega$ ,  $Y_\Omega$  and  $L_\Omega$  are related by the same *identities* as in the classical theory. Nevertheless, one may wonder whether those can be replaced by some *double-sided estimates* which do not depend on geometric properties of  $(\Omega; a, b, c, d)$ . One of the main results of our paper is Theorem 7.1 which gives a positive answer to this question. Namely, it says that, provided  $L_\Omega \geq \text{const}$ , one has

$$Z_\Omega \asymp Y_\Omega \quad \text{and} \quad \log(1 + Y_\Omega^{-1}) \asymp L_\Omega,$$

*uniformly* over all possible discrete quadrilaterals. Note that we use discrete cross-ratio  $Y_\Omega$  as an intermediary that allows us to relate “analytic” partition function  $Z_\Omega$  and “geometric” extremal length  $L_\Omega$  in a way which is very similar to the classical setup.

In order to illustrate a potential of the toolbox developed in our paper, we include two applications of a different kind. The first, given in Sect. 5, is a “surgery technique” for discrete quadrilaterals which is important for the fine analysis of interfaces in the critical Ising model, see [CDH12]. Namely, we show that it is always possible to cut  $\Omega$  along some family of slits  $L_k$  into two parts  $\Omega'_k$  and  $\Omega''_k$  (containing  $[ab]_\Omega$  and  $[cd]_\Omega$ ,

respectively) so that, for any  $k$ , one has

$$Z_\Omega \asymp Z_{\Omega'_k}([ab]_\Omega; L_k) \cdot Z_{\Omega''_k}(L_k; [cd]_\Omega) \quad \text{and} \quad Z_{\Omega'_k}([ab]_\Omega; L_k) \asymp k \cdot Z_{\Omega''_k}(L_k; [cd]_\Omega)$$

(see Theorem 5.1 for details). Using a discrete cross-ratios technique, we prove this, quite natural from a geometric point of view, result without any reference to the actual geometry of  $\Omega$ . As always in our paper, double-sided estimates given above are uniform with respect to  $(\Omega; a, b, c, d)$  and  $k$ .

Another application, given in Sect. 7, allows one to control a discrete harmonic measure  $\omega_{\text{disc}} := \omega_\Omega(u; [ab]_\Omega)$  of a “far” boundary arc  $[ab]_\Omega \subset \partial\Omega$  via an appropriate discrete extremal length  $L_{\text{disc}}$  in  $\Omega$  (see Sect. 7 and Theorem 7.8 for details). This should be considered as an analogue of the famous Ahlfors-Beurling-Carleman estimate (see [GM05, Theorem IV.5.2] and [GM05, p.150] for historical notes). Again, we get a uniform double-sided bound which, as a byproduct, imply that

$$\log(1 + \omega_{\text{disc}}^{-1}) \asymp L_{\text{disc}} \asymp L_{\text{cont}} \asymp \log(1 + \omega_{\text{cont}}^{-1})$$

*uniformly* over all possible configurations  $(\Omega; u, a, b)$ , where  $\omega_{\text{cont}}$  denotes the classical harmonic measure of the boundary arc  $[ab]$  seen from  $u$  in the polygonal representation of  $\Omega$  (see Corollary 7.9 for details). Note that results of this sort seem to be hardly available by any kind of coupling arguments. Indeed, dealing with thin fiords we are mostly focused on exponentially rare events for both discrete random walks and the (continuous) Brownian motion which are highly sensitive to widths of those fiords.

**1.3. Organization of the paper.** In **Section 2** we list several needed (rather mild) assumptions on the underlying weighted graph  $\Gamma$  and two important Assumptions (S) and (T) for the random walk on  $\Gamma$  (namely, uniform estimates for hitting probabilities and the expected exit time for discrete approximations of *Euclidean discs*). Then, we use (S) and (T) as a “black box”, not discussing what one should ask about  $\Gamma$  to guarantee these properties. We also fix the notation for discrete domains  $\Omega$ , introduce the partition functions  $Z_\Omega$  of the simple random walk in  $\Omega$ , and discuss its relation with the standard notions of discrete harmonic measure and discrete Green’s function.

**Section 3** is devoted to a uniform (up to absolute multiplicative constants) *factorization of the three-point partition function*  $Z_\Omega(a; [bc]_\Omega)$  via two-point functions  $Z_\Omega(a; b)$ ,  $Z_\Omega(a; c)$  and  $Z_\Omega(b; c)$ . Namely, we prove that (see Theorem 3.5)

$$Z_\Omega(a; [bc]_\Omega) \asymp [Z_\Omega(a; b)Z_\Omega(a; c) / Z_\Omega(b; c)]^{\frac{1}{2}}$$

uniformly over all configurations  $(\Omega; a, b, c)$ . This is a cornerstone of our paper and the only one place where we involve some geometric considerations in the proofs.

In **Section 4**, we introduce *discrete cross-ratios*  $X_\Omega, Y_\Omega$  for a simply connected domain  $\Omega$  with four marked boundary points  $a, b, c, d$  (see Definition 4.3) and deduce from Theorem 3.5 several double-sided estimates relating  $X_\Omega, Y_\Omega$  and  $Z_\Omega$ . In particular, we prove that  $X_\Omega^{-1} \asymp 1 + Y_\Omega^{-1}$  (see Proposition 4.5, this is an analogue of the well known identity for classical cross-ratios) and  $Z_\Omega \asymp \log(1 + Y_\Omega)$  (see Theorem 4.8, this is a precursor of the exponential-type estimate relating  $Z_\Omega$  and  $L_\Omega$ ).

**Section 5** is independent of the rest of the paper. It shows how one can use Theorem 3.5 and discrete cross-ratios introduced in Sect. 4 in order to build a sort of “*surgery technique*” which allows one to effectively “decouple” dependence  $Z_\Omega$  of the boundary arcs  $[ab]_\Omega$  and  $[cd]_\Omega$  by finding nice discrete cross-cuts in  $\Omega$ .

In **Section 6**, the notion of *discrete extremal length*  $L_\Omega([ab]; [cd])$  comes into play. We recall its definition and prove that  $L_\Omega$  is always uniformly comparable to its continuous counterpart – extremal length of the family of curves connecting  $[ab]$  and  $[cd]$  in the polygonal representation of  $\Omega$  (see Fig. 1B). In particular, this fact implies the very important *duality estimate* for discrete extremal lengths (see Corollary 6.3). We also prove some simple inequalities relating  $Z_\Omega$  and  $L_\Omega^{-1}$  (see Proposition 6.6).

**Section 7** summarizes all estimates for  $Y_\Omega$ ,  $Z_\Omega$  and  $L_\Omega$  into the single Theorem 7.1 which is the culmination of our paper. Then, we show how to fit a *discrete harmonic measure*  $\omega_\Omega(u; [ab]_\Omega)$  in this context (as  $\Omega \setminus \{u\}$  is not simply connected, a reduction similar to [GM05, p. 144] is needed). The main result (double-sided estimate of  $\omega_\Omega(u; [ab]_\Omega)$  via an appropriate extremal length) is given by Theorem 7.8. As a simple byproduct, we prove Corollary 7.9 which says that the logarithm of a discrete harmonic measure is uniformly comparable to its continuous counterpart.

Finally, in **Appendix** we derive several needed facts of discrete potential theory (elliptic Harnack principle, weak Beurling estimate, and pointwise estimates for Green’s function) from Assumptions (S),(T). In some sense, we consider (S),(T) as a “pointe de la jonction”: being formulated for simplest possible domains (discrete approximations of Euclidean discs), they can be obtained by some specific means for various graphs, providing a basement for our toolbox which is more adapted for very rough  $\Omega$ ’s.

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## 2. NOTATION, ASSUMPTIONS AND PRELIMINARIES

**2.1. Graph notation and assumptions.** Throughout this paper we work with an infinite undirected weighted planar graph  $(\Gamma; E^\Gamma)$  embedded into a complex plane  $\mathbb{C}$  so that all its edges are straight segments (see Fig. 1) which is assumed to satisfy the list of assumptions (a)–(d) given below as well as Assumptions (S) and (T) formulated in Sect. 2.3. The notation  $\Gamma \subset \mathbb{C}$  is fixed for the set of vertices which are understood as points in the complex plane (so  $|u - v|$  means the Euclidean distance between  $u, v \in \Gamma$ ), and  $E^\Gamma$  denote the corresponding set of edges. Each edge  $e \in E^\Gamma$  is equipped with the positive *weight*  $w_e$ . Note that, in general, these weights are *not* related to the way how  $\Gamma$  is embedded into  $\mathbb{C}$ . We assume that  $\Gamma$  satisfy

- **(a) uniformly bounded degree assumption:** there is a constant  $\nu_0 > 0$  such that  $\mu_v := \sum_{(vv') \in E^\Gamma} w_{vv'} \leq \nu_0$  for all  $v \in \Gamma$  and  $w_e \geq \nu_0^{-1}$  for all  $e \in E^\Gamma$ .

Clearly, this is equivalent to say that all edge weights  $w_e$  are uniformly bounded away from 0 and  $\infty$  and all degrees of vertices of  $\Gamma$  are uniformly bounded as well. We then

denote *random walk transition probabilities* by

$$\varpi_{vv'} := \frac{w_{vv'}}{\mu_v} = \frac{w_{vv'}}{\sum_{(vv') \in E^\Gamma} w_{vv'}}. \quad (2.1)$$

Note that  $\varpi_{vv'}$  are uniformly bounded below by  $\nu_0^{-2} > 0$  and  $\sum_{(vv') \in E^\Gamma} \varpi_{vv'} = 1$ ,  $v \in \Gamma$ .

We now pass to the way how  $\Gamma$  is *embedded* into  $\mathbb{C}$ . We assume that

- **(b)** *there are no “flat” angles*: there exists a constant  $\eta_0 > 0$  such that all angles between neighboring edges at any vertex  $v \in \Gamma$  do not exceed  $\pi - \eta_0$  (this also implies that all degrees of *faces* of  $\Gamma$  are uniformly bounded by  $2\pi/\eta_0$ );
- **(c)** *edge lengths are locally comparable*: there exists a constant  $\rho_0 \geq 1$  such that, for any vertex  $v \in \Gamma$ , one has

$$\max_{(vv') \in E^\Gamma} |v' - v| \leq \rho_0 r_v, \quad \text{where} \quad r_v := \min_{(vv') \in E^\Gamma} |v' - v| \quad (2.2)$$

(below we sometimes call  $r_v$  the *local scale size*);

- **(d)**  $\Gamma$  is “*quantitatively locally finite*”: for any  $\rho \geq 1$  there exists some constant  $\nu(\rho) > 0$  such that  $\#\{v \in \Gamma : |v - u| \leq \rho r_u\} \leq \nu(\rho)$ , uniformly over all  $u \in \Gamma$ .

Roughly speaking, the last assumption prevents us from embedding  $\Gamma$  so that vertices almost accumulate to a finite point, giving a uniform bound of their number in discs of radii comparable to the local scale size. Note that we do *not* assume that  $\nu(\rho)$  grows quadratically in  $\rho$ : the local scale size *can* vary from place to place but “not too fast”.

**Remark 2.1.** It is easy to see that there exists a constant  $\varepsilon_0 > 0$  (depending on constants from (a)–(d) only) such that, for any  $u, v \in \Gamma$ , the *length*  $\sum_{k=0}^{n-1} |u_{k+1} - u_k|$  of the shortest discrete path  $(u_0 u_1 \dots u_n)$ ,  $(u_k u_{k+1}) \in E^\Gamma$ , connecting  $u = u_0$  and  $v = u_n$  is bounded by  $\varepsilon_0^{-1} \cdot |u - v|$  (to construct a short path, use (b) to do the next step from  $u_k$  towards  $v$ , if  $r_{u_k} \ll |u_k - v|$ , and (d), when reaching  $u_k$  such that  $r_{u_k} \asymp |u_k - v|$ ).

**2.2. Bounded discrete domains.** We start with a definition of a (bounded) discrete domain  $\Omega$  (see Fig. 1). Let  $(V^\Omega; E_{\text{int}}^\Omega)$  be a bounded *connected* subgraph of  $(\Gamma; E^\Gamma)$ . In order to make the presentation simpler and do not overload the notation, we always assume that  $(ab) \in E_{\text{int}}^\Omega$  for any two neighboring (in  $\Gamma$ ) vertices  $a, b \in V^\Omega$  (one can easily remove this assumption, if necessary). Denote by  $E_{\text{bd}}^\Omega$  the set of all *oriented* edges  $(a_{\text{int}}a) \notin E_{\text{int}}^\Omega$  such that  $a_{\text{int}} \in V^\Omega$  (and  $a \notin V^\Omega$ ). We set  $\Omega := \text{Int } \Omega \cup \partial\Omega$ , where

$$\text{Int } \Omega := V^\Omega, \quad \partial\Omega := \{(a; (a_{\text{int}}a)) : (a_{\text{int}}a) \in E_{\text{bd}}^\Omega\}.$$

Formally, the boundary  $\partial\Omega$  of a discrete domain  $\Omega$  should be treated as the set of oriented edges  $(a_{\text{int}}a)$ , but we usually identify it with the set of corresponding vertices  $a$ , and think about  $\text{Int } \Omega$  and  $\partial\Omega$  as subsets of  $\Gamma$ , if no confusion arises.

For a given vertex  $u \in \Gamma$  and  $r \geq r_u$ , we denote by  $\mathbf{B}_r^\Gamma(u)$  the **discrete disc** of radius  $r$  around  $u$ . Namely,  $\text{Int } \mathbf{B}_r^\Gamma(u)$  is the set of all vertices  $v \in \Gamma$  lying in the connected component of  $\Gamma \cap \{v : |v - u| < r\}$  containing  $u$  (e.g.,  $\text{Int } \mathbf{B}_{r_u}^\Gamma(u) = \{u\}$ ), and  $\partial\mathbf{B}_r^\Gamma(u)$  is the set of their neighbors, see Fig. 1A).

**Remark 2.2.** It is worth to mention two useful corollaries of assumptions (a)–(d):

- (i) There exists a constant  $\varepsilon_0 > 0$  such that, for any  $u \in \Gamma$  and  $r \geq r_u$ , one has  $\Gamma \cap \{v : |v - u| \leq \varepsilon_0 r\} \subset \text{Int } \mathbf{B}_r^\Gamma(u)$  (this immediately follows from Remark 2.1).

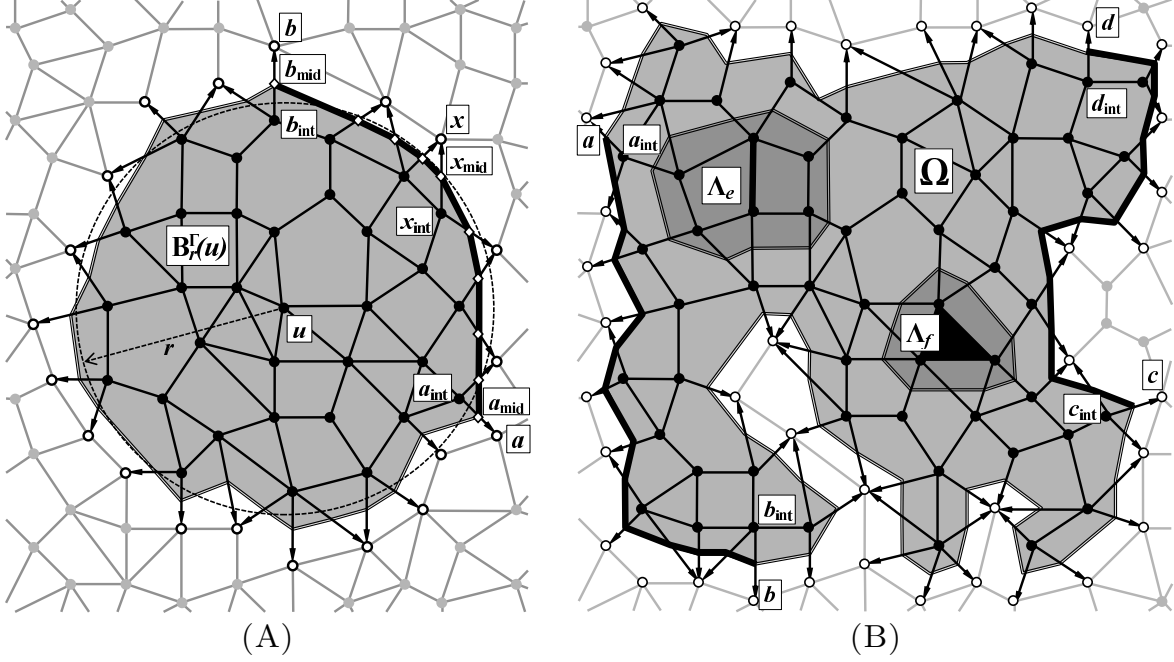


FIGURE 1. (A) A discrete disc  $B_r^\Gamma(u)$  and its polygonal representation (dashed). The inner vertices are colored black, the boundary ones are white. For all boundary edges  $(x_{int}x)$  on the counterclockwise boundary arc  $[ab]_{B_r^\Gamma(u)} \subset \partial B_r^\Gamma(u)$  the corresponding midpoints  $x_{mid}$  are marked. (B) An example of a discrete simply connected domain  $\Omega$  with two marked boundary arcs  $[ab]_\Omega$  and  $[cd]_\Omega$ . Inside of  $\Omega$ , the notation used in the proof of Proposition 6.2 is shown: the polygonal representations of the neighborhoods  $\Lambda_e$  and  $\Lambda_f$  of an edge and a face.

(ii) For any given  $\delta > 0$ , there exists a positive constant  $\nu(2\delta^{-1}) > 0$  such that, for all  $u \in \Gamma$  and  $r \geq r_u$ , either  $r_v \leq \delta r$  for all vertices  $v \in \text{Int } B_r^\Gamma(u)$ , or  $\#B_r^\Gamma(u) \leq \nu(2\delta^{-1})$  (this is a direct consequence of (d) as  $B_r^\Gamma(u) \subset B_{2r}^\Gamma(v)$ ).

We say that a discrete domain  $\Omega$  is *simply connected*, if for any cycle in  $E_{int}^\Omega$ , all edges of  $\Gamma$  surrounded by this cycle also belong to  $E_{int}^\Omega$ . If  $\Omega$  is simply connected, then its boundary vertices (or, more precisely, boundary edges) are naturally cyclicly ordered, exactly as in the continuous setting. For two boundary vertices  $a, b \in \partial\Omega$  of a simply connected  $\Omega$  we denote a **boundary arc**  $[ab]_\Omega \subset \partial\Omega$  as the set of all boundary vertices lying between  $a$  and  $b$  (including those two) when one goes along  $\partial\Omega$  in the *counterclockwise* direction, see Fig. 1A (so,  $[ab]_\Omega \cup [ba]_\Omega = \partial\Omega$  and  $[ab]_\Omega \cap [ba]_\Omega = \{a, b\}$ ). We also use the notation  $[ab]_\Omega := [ab]_\Omega \setminus \{b\}$ ,  $(ab)_\Omega := [ab]_\Omega \setminus \{a\}$  etc.

For a real function  $H : \Omega \rightarrow \mathbb{R}$  defined in  $\Omega$ , we define its *discrete Laplacian* by

$$[\Delta H](v) := \sum_{(vv') \in E^\Gamma} \varpi_{vv'}(H(v') - H(v)), \quad v \in \text{Int } \Omega,$$

where the sum is taken over all neighbors of  $v$ , and  $\varpi_{vv'}$  are given by (2.1). We say that  $H$  is *discrete harmonic in  $\Omega$* , if  $[\Delta H](v) = 0$  at all inner vertices  $v \in \text{Int } \Omega$ . Note that, if  $H \geq 0$  and  $[\Delta H](v) = 0$ , then  $H(v')/H(v) \leq \varpi_{vv'}^{-1} \leq \nu_0^2$  (see Sect. 2.1).

Below we often use two basic notions of discrete potential theory. The first is the discrete **harmonic measure**  $\omega_\Omega(\mathbf{u}; E)$  of a boundary set  $E \subset \partial\Omega$  seen from an (inner) point  $u \in \Omega$ . It can be defined as the unique function which is discrete harmonic in  $\Omega$  and coincides with  $\mathbb{1}_E(\cdot)$  on  $\partial\Omega$ . Note that  $\omega_\Omega(u; E)$  has a simple probabilistic meaning: it is equal to the probability of the event that the random walk having transition probabilities (2.1) started at  $u$  firstly hits  $\partial\Omega$  on  $E$ . The second is the (positive) **Green's function**  $G_\Omega(\mathbf{v}; \mathbf{u})$ . It is the unique function which is discrete harmonic everywhere in  $\Omega$  except at  $u$ , vanishes on the boundary  $\partial\Omega$ , and such that

$$[\Delta G_\Omega(\cdot; u)](u) = -\mu_u^{-1}.$$

From the probabilistic point of view,  $G_\Omega(v; u)$  is equal to the expected number of visits at  $u$  (divided by  $\mu_u$ ) of the random walk started at  $v$  and stopped when reaching  $\partial\Omega$ . Note that  $G_\Omega$  is symmetric, i.e.,  $G_\Omega(u; v) \equiv G_\Omega(v; u)$  (e.g., see Remark 2.6(ii)).

**2.3. Random walk assumptions.** We base our paper on two important assumptions for the random walk on  $\Gamma$  having transition probabilities (2.1). Namely, below we assume that two following properties hold true:

**Assumption (S) (“space”).** *There exist two positive constants  $\eta_0, c_0 > 0$  such that, uniformly over all  $u \in \Gamma$ ,  $r \geq r_u$  and  $\theta \in [0, 2\pi]$ , one has*

$$\omega_{B_r^\Gamma(u)}(u; \{a \in \partial B_r^\Gamma(v) : \arg(a - u) \in [\theta, \theta + (\pi - \eta_0)]\}) \geq c_0.$$

In other words, we assume that there are no exceptional directions: the random walk started at the center of any discrete disc  $B_r^\Gamma(u)$  can exit this disc through any given boundary arc of the angle  $\pi - \eta_0$  with probability uniformly bounded away from 0.

**Assumption (T) (“time”).** *There exist two positive constants  $c_0, C_0 > 0$  such that, uniformly over all  $u \in \Gamma$  and  $r \geq r_u$ , the following is fulfilled:*

$$c_0 r^2 \leq \sum_{v \in \text{Int } B_r^\Gamma(u)} r_v^2 G_{B_r^\Gamma(u)}(v; u) \leq C_0 r^2.$$

Despite (T) is formulated in terms of discrete harmonic functions only (which do not encounter any particular time parametrization of the random walk), it is natural to mention the following interpretation: let us consider some time parametrization such that the (expected) time spent by the walk at a vertex  $v$  before it jumps is of order  $r_v^2$  (recall that local scales  $r_v$  can be quite different for different  $v$ 's, if  $r \gg r_u$ ). Then we ask the expected time spent in a discrete disc  $B_r(u)$  by the random walk started at  $u$  before it hits  $\partial B_r(u)$  to be of order  $r^2$ , uniformly over all discrete discs. Note that, since  $\sum_{v \in \text{Int } B_r(u)} r_v^2$  is comparable to  $r^2$  due to (b), (c) and simple geometric reasons, (T) would follow from the following “analytic” claim: Green's function  $G_{B_r(u)}(v; u)$  is of order 1 for those  $v \in B_r(u)$  which are away from  $u$  and  $\partial B_r(u)$  and the total contribution of  $v$ 's lying close to  $u$  (where  $G_{B_r(u)}$  blows up) is bounded, see also Lemma A.5.

**Remark 2.3.** From now on, we think about all constants  $\nu_0, \eta_0, \rho_0$  etc. used in assumptions (a)–(d) and (S), (T) as *fixed once forever*. Thus, below we say, e.g., “with some uniform constants  $\text{const}_1$  and  $\text{const}_2$ ” meaning that  $\text{const}_{1,2}$  may, in general, depend on  $\nu_0, \eta_0, \rho_0$  etc., but are independent of all other parameters involved (like domain shape, location of boundary points or particular graph structure).



**Remark 2.4.** In this paper, we use Assumptions (S) and (T) as a “black box”, not intending to discuss what one should ask for the graph  $\Gamma$  and its embedding into  $\mathbb{C}$  in order to guarantee them. Nevertheless, note that the following natural question arises:

*whether assumptions (a)–(d) are already sufficient for (S) and (T) or not?*

It is worth to mention that, if all lengths of edges  $e \in E^\Gamma$  are uniformly comparable to each other, then (S) and (T) for the *simple* random walk on  $\Gamma$  can be derived from the Parabolic Harnack Inequality technique (e.g., see discussion in [Koz07, Sect. 2.1]). An easier example when (S) and (T) can be obtained due to nice “local approximation properties” is given by some special random walks on isoradial graphs, see [CS11].

**2.4. Partition function of the random walk in a discrete domain  $\Omega$ .** The following notation generalizes both discrete harmonic measure and Green’s function.

**Definition 2.5.** Let  $\Omega \subset \Gamma$  be a bounded discrete domain and  $x, y \in \Omega$ . We denote by  $Z_\Omega(x; y)$  the **partition function of the random walk joining  $x$  and  $y$  inside  $\Omega$** . Namely,

$$Z_\Omega(x; y) := \sum_{\gamma \in S_\Omega(x; y)} w(\gamma), \quad w(\gamma) := \frac{\prod_{k=0}^{n(\gamma)-1} w_{u_k u_{k+1}}}{\prod_{k=0}^{n(\gamma)} \mu_k} = \mu_{n(\gamma)}^{-1} \prod_{k=0}^{n(\gamma)-1} \varpi_{u_k u_{k+1}}, \quad (2.3)$$

where  $S_\Omega(x; y) = \{\gamma = (u_0 \sim u_1 \sim \dots \sim u_{n(\gamma)}) : u_0 = x; u_1, \dots, u_{n-1} \in \text{Int } \Omega; u_n = y\}$  is the set of all nearest-neighbor paths connecting  $x$  and  $y$  inside  $\Omega$ . Further, for  $A, B \subset \Omega$ , we set

$$Z_\Omega(A; B) := \sum_{x \in A, y \in B} Z_\Omega(x; y),$$

and by  $\mathbf{RW}_\Omega(\mathbf{A}; \mathbf{B})$  we denote a random nearest-neighbor path  $\gamma$  chosen from the set  $S_\Omega(A; B) := \cup_{x \in A, y \in B} S_\Omega(x; y)$  with probabilities proportional to  $w(\gamma)$ .

**Remark 2.6.** It is easy to see that

- (i) if  $u \in \text{Int } \Omega$  and  $b \in \partial\Omega$ , then  $Z_\Omega(u; b) = \mu_b^{-1} \omega_\Omega(u; b)$ ;
- (ii) if both  $u, v \in \text{Int } \Omega$ , then  $Z_\Omega(v; u) = G_\Omega(v; u)$ .

*Proof.* (i) Focusing on the first step of  $\gamma \in S_\Omega(u; b)$  in (2.3), one immediately concludes that the function

$$H(u) := \begin{cases} Z_\Omega(u; b), & u \in \text{Int } \Omega, \\ \mu_b^{-1} \mathbb{I}[u = b], & u \in \partial\Omega, \end{cases}$$

is discrete harmonic in  $\Omega$  and coincides with  $\mu_b^{-1} \omega_\Omega(\cdot; b)$  on the boundary  $\partial\Omega$ . Thus,  $Z_\Omega(u; b) = H(u) = \omega_\Omega(u; b)$  for all  $u \in \text{Int } \Omega$ .

(ii) As above, it immediately follows from (2.3) that the function

$$H(v) := \begin{cases} Z_\Omega(v; u), & v \in \text{Int } \Omega, \\ 0, & v \in \partial\Omega, \end{cases}$$

is discrete harmonic everywhere in  $\Omega$  except at  $u$ , and

$$H(u) = \mu_u^{-1} + \sum_{(uu') \in E^\Gamma} \varpi_{uu'} H(u'),$$

where the first term  $\mu_u^{-1}$  corresponds to the trivial trajectory consisting of a single point  $u$ . Thus,  $[\Delta H](u) = -\mu_u^{-1}$  and  $Z_\Omega(v; u) = H(v) = G_\Omega(v; u)$  for all  $v \in \text{Int } \Omega$ .  $\square$

### 3. FACTORIZATION THEOREM FOR THE THREE-POINT FUNCTION $Z_\Omega(a; [bc]_\Omega)$

The main result of this Section is Theorem 3.5. It deals with a simply connected discrete domain  $\Omega$  and three marked boundary points  $a, b, c \in \partial\Omega$  (no assumptions about actual geometry of  $(\Omega; a, b, c)$  are used) and provides a uniform up-to-constant factorization of the three-point function  $Z_\Omega(a; [bc]_\Omega)$  via  $Z_\Omega(a; b)$ ,  $Z_\Omega(a; c)$  and  $Z_\Omega(b; c)$ . Actually, our proof is based on a factorization of the latter two-point functions via some inner point  $u \in \text{Int } \Omega$  which is “not too close” to any of the boundary arcs  $[ab]_\Omega$ ,  $[bc]_\Omega$  and  $[ca]_\Omega$ . Thus, our strategy to prove Theorem 3.5 can be described as follows:

- prove that the ratio  $Z_\Omega(a; u)Z_\Omega(u; b)/Z_\Omega(a; b)$  is uniformly comparable with the probability of the event that  $\text{RW}_\Omega(a; b)$  passes “not very far” from  $u$  (namely, at distance less than  $\frac{1}{4} \text{dist}(u; \partial\Omega)$ ) – this is done in Proposition 3.1;
- prove that this probability is bounded below, if  $u$  is “not too close” to any of boundary arcs  $[ab]_\Omega$  and  $[ba]_\Omega$  – this is done in Lemma 3.2 and Proposition 3.3;
- find an inner vertex  $u$  which is “not too close” to any of  $[ab]_\Omega$ ,  $[bc]_\Omega$  and  $[ca]_\Omega$  – this is done in Lemma 3.4 – and factorize all  $Z_\Omega$ ’s using this  $u$ .

Below we use the following notation. For a discrete domain  $\Omega$  and  $u \in \text{Int } \Omega$ , let

$$d_\Omega(u) := \text{dist}(u; \partial\Omega) = \min_{b \in \partial\Omega} |u - b|, \quad B_\Omega(u) := B_{d_\Omega(u)/4}^\Gamma(u). \quad (3.1)$$

Recall that (3.1) means  $\text{Int } B_\Omega(u) = \{v \in \Gamma : |v - u| < \frac{1}{4} \text{dist}(u; \partial\Omega)\} \subset \text{Int } \Omega$  (or, more accurate, a connected component of this set) and  $\partial B_\Omega(u) \subset \Omega$  is the set of all vertices neighboring to  $\text{Int } B_\Omega(u)$ . We also generalize notation (2.3) in the following way: for a given subdomain  $U \subset \Omega$  and a random walk path  $\gamma = (u_0 \sim u_1 \sim \dots \sim u_{n(\gamma)})$ , we set

$$T_U(\gamma) := \sum_{k=0}^{n(\gamma)} r_{u_k}^2 \mathbb{1}[u_k \in \text{Int } U].$$

Then, for  $A, B \subset \Omega$ , we define

$$Z_\Omega[\mathbf{T}_U](\mathbf{A}; \mathbf{B}) := \sum_{\gamma \in S_\Omega(A; B)} w(\gamma) T_U(\gamma).$$

It is worth to note that

$$\frac{Z_\Omega[\mathbf{T}_U](A; B)}{Z_\Omega(A; B)} = \mathbb{E}[T_U(\text{RW}_\Omega(A; B))]$$

is the expected time spent in  $U$  by a (properly parameterized) random walk  $\text{RW}_\Omega(A; B)$ .

**Proposition 3.1.** *Let  $\Omega$  be a simply connected discrete domain,  $a, b \in \partial\Omega$ , and  $u \in \text{Int } \Omega$ . Then, the following double-sided estimate is fulfilled:*

$$\frac{Z_\Omega(u; a)Z_\Omega(u; b)}{Z_\Omega(a; b)} \asymp \mathbb{P}[\text{RW}_\Omega(a; b) \cap \text{Int } B_\Omega(u) \neq \emptyset],$$

with some absolute (i.e., independent of  $\Omega, a, b, u$ ) constants.

*Proof.* Recall that both functions  $Z_\Omega(\cdot; a) = \mu_a^{-1} \omega_\Omega(\cdot; a)$  and  $Z_\Omega(\cdot; b) = \mu_b^{-1} \omega_\Omega(\cdot; b)$  are discrete harmonic and positive inside  $\Omega$ . Therefore, Harnack’s Principle (see Lemma A.2) gives

$$Z_\Omega(u; a)Z_\Omega(u; b) \asymp \frac{\sum_{v \in \text{Int } B_\Omega(u)} r_v^2 Z_\Omega(v; a)Z_\Omega(v; b)}{\sum_{v \in \text{Int } B_\Omega(u)} r_v^2} \quad (3.2)$$

and  $\sum_{v \in \text{Int } B_\Omega(u)} r_v^2 \asymp (d_\Omega(u))^2$  due to simple geometric reasons.

Joining two random walk paths  $\gamma_{av}$  (from  $a$  to  $v$ ) and  $\gamma_{vb}$  (from  $v$  to  $b$ ), and taking into account  $w(\gamma_{av}\gamma_{vb}) = \mu_v \cdot w(\gamma_{av})w(\gamma_{vb}) \asymp w(\gamma_{av})w(\gamma_{vb})$ , it is easy to see that

$$\sum_{v \in \text{Int } B_\Omega(u)} r_v^2 Z_\Omega(v; a) Z_\Omega(v; b) \asymp Z_\Omega[T_{B_\Omega(u)}](a; b) \quad (3.3)$$

(indeed, each of the vertices  $u_k \in \text{RW}_\Omega(a; b)$  contributing to  $T_{B_\Omega(u)}$  can be chosen as  $v$  in order to split  $\text{RW}_\Omega(a; b)$  into two halves  $\gamma_{av}$  and  $\gamma_{vb}$ ).

Further, let  $w$  denote the *first* vertex  $u_k \in \text{Int } B_\Omega(u)$  of  $\text{RW}_\Omega(a; b)$ , if such a vertex exists. Since in the right-hand side of (3.3) we do not count those paths which don't intersect  $B_\Omega(u)$ , by splitting  $\text{RW}_\Omega(a; b)$  into two halves at  $w$ , (3.3) can be rewritten as

$$Z_\Omega[T_{B_\Omega(u)}](a; b) \asymp \sum_{w \in \text{Int } B_\Omega(u)} Z_{\Omega \setminus B_\Omega(u)}(a; w) Z_\Omega[T_{B_\Omega(u)}](w; b), \quad (3.4)$$

where a (generally, doubly connected) discrete domain  $\Omega' := \Omega \setminus B_\Omega(u)$  should be understood so that  $\text{Int } \Omega' = \text{Int } \Omega \setminus \text{Int } B_\Omega(u)$ . It immediately follows from our definition of  $Z[T]$  and Harnack's Principle applied to the discrete harmonic function  $Z_\Omega(\cdot; b)$  that

$$Z_\Omega[T_{B_\Omega(u)}](w; b) \asymp \sum_{v \in \text{Int } B_\Omega(u)} r_v^2 Z_\Omega(w; v) Z_\Omega(v; b) \asymp \sum_{v \in \text{Int } B_\Omega(u)} r_v^2 Z_\Omega(w; v) \cdot Z_\Omega(w; b) \quad (3.5)$$

(indeed, for each  $u_k$  contributing to  $T_{B_\Omega(u)}(\text{RW}_\Omega(w; b)) = \sum_{k=0}^{n(\gamma)} r_{u_k}^2 \mathbb{1}[u_k \in \text{Int } B_\Omega(u)]$ , split the random path  $\text{RW}_\Omega(w; b)$  into two halves  $\gamma_{wv}, \gamma_{vb}$  at the point  $v = u_k$  and use the up-to-constant multiplicativity  $w(\gamma_{wv}\gamma_{vb}) \asymp w(\gamma_{wv})w(\gamma_{vb})$  once more). Moreover, using Assumption (T), it is easy to conclude that

$$\sum_{v \in \text{Int } B_\Omega(u)} r_v^2 Z_\Omega(w; v) = \sum_{v \in \text{Int } B_\Omega(u)} r_v^2 G_\Omega(v; w) \asymp (d_\Omega(u))^2 \quad \text{for any } w \in \text{Int } B_\Omega(u).$$

(see Lemma A.5 and Remark A.6). Combining this with (3.2)–(3.5), one obtains

$$\begin{aligned} \frac{Z_\Omega(u; a) Z_\Omega(u; b)}{Z_\Omega(a; b)} &\asymp (d_\Omega(u))^{-2} \frac{Z_\Omega[T_{B_\Omega(u)}](a; b)}{Z_\Omega(a; b)} \\ &\asymp \frac{\sum_{w \in \text{Int } B_\Omega(u)} Z_{\Omega \setminus B_\Omega(u)}(a; w) Z_\Omega(w; b)}{Z_\Omega(a; b)}. \end{aligned} \quad (3.6)$$

Finally, the numerator can be rewritten as

$$\sum_{w \in \text{Int } B_\Omega(u)} Z_{\Omega \setminus B_\Omega(u)}(a; w) Z_\Omega(w; b) \asymp \sum_{\gamma \in S_\Omega(a; b): \gamma \cap \text{Int } B_\Omega(u) \neq \emptyset} w(\gamma)$$

(as above, denote by  $w$  the first vertex  $u_k \in \text{Int } B_\Omega(u)$  of  $\gamma$ , if such a vertex exists). Thus, (3.6) is comparable to the probability of the event  $\text{RW}_\Omega(a; b) \cap B_\Omega(u) \neq \emptyset$ .  $\square$

Below we use the shorter notation

$$\mathbb{P}_\Omega^{a, b}[\mathbf{B}_\Omega(u)] := \mathbb{P}[\text{RW}_\Omega(a; b) \cap \text{Int } B_\Omega(u) \neq \emptyset]$$

for the right-hand side of (3.1). Also, for  $u \in \Omega$  and  $r \geq r_u$ , we denote by  $B_r^\Omega(u)$  the connected component of  $\Omega \cap B_r^\Gamma(u)$  containing  $u$  (and call it *r-neighborhood in  $\Omega$  of  $u$* ).

**Lemma 3.2.** *Let  $\Omega$  be a simply connected discrete domain,  $a, b \in \partial\Omega$ , and  $u \in \text{Int } \Omega$ . Let  $L_u$  denote the shortest discrete path running from  $u$  to  $\partial\Omega$ ,  $d := \text{Length}(L_u)$  (recall that  $d \leq \varepsilon_0^{-1} d_\Omega(u)$ , see Remark 2.1), and  $a, b \notin B_{2d}^\Omega(u)$ . Then,*

$$\mathbb{P}[\text{RW}_\Omega(a; b) \cap L_u \neq \emptyset] \leq \text{const} \cdot \mathbb{P}_\Omega^{a,b}[B_\Omega(u)].$$

*Proof.* Let  $u = u_0, u_1, \dots, u_n \in \Omega$  be defined inductively by the following rule:  $u_{k+1} \in L_u$  is the first vertex on  $L_u$  which doesn't belong to  $\text{Int } B_\Omega(u_k)$  (thus, each  $u_{k+1} \in \partial B_\Omega(u_k)$  and, finally,  $x := u_n \in \partial\Omega$ ). Further, let  $u_m$  be the first of those  $u_k$ 's such that the rest of the path  $L_u$  is contained in the  $\frac{1}{2}d_\Omega(u)$ -neighborhood (in  $\Omega$ ) of the ending point  $x$ .

Since for any  $v \in L_u$  the distance from  $v$  to  $\partial\Omega$  along  $L_u$  is comparable to  $d_\Omega(v)$ , one concludes that  $d_\Omega(u_k)$  (or, equivalently, distances along  $L_u$ ) decreases exponentially as  $k$  grows. In particular, this implies  $m \leq \text{const}$ . Let  $L'_u$  and  $L''_u$  denote the parts of  $L_u$  from  $u$  to  $u_m$  and from  $u_m$  to  $\partial\Omega$ , respectively. Harnack's principle implies

$$H(u) \asymp H(u_1) \asymp \dots \asymp H(u_m)$$

and so  $H \asymp H(u)$  everywhere on  $L'_u$  (with some absolute constants), for each of two functions  $H = Z_\Omega(\cdot; a)$  and  $H = Z_\Omega(\cdot; b)$ . Therefore,

$$\begin{aligned} \mathbb{P}[\text{RW}_\Omega(a; b) \cap L'_u \neq \emptyset] &\leq \sum_{k=0}^m \mathbb{P}_\Omega^{a,b}[B_\Omega(u_k)] \asymp \sum_{k=0}^m \frac{Z_\Omega(u_k; a)Z_\Omega(u_k; b)}{Z_\Omega(a; b)} \\ &\leq \text{const} \cdot \frac{Z_\Omega(u; a)Z_\Omega(u; b)}{Z_\Omega(a; b)} \asymp \mathbb{P}_\Omega^{a,b}[B_\Omega(u)]. \end{aligned} \quad (3.7)$$

Due to Remark 2.2, without loss of generality, we may assume that  $r_v \leq \delta_0 d_\Omega(u)$  for all  $v \in B_{d_\Omega(u)}^\Omega(x)$ , where  $\delta_0$  is chosen small enough independently of  $\Omega$  and  $u$  (otherwise,  $L''_u$  contains only a uniformly bounded number of vertices and the proof can be easily finished similarly to (3.7)). Then, it can be easily derived from (S) that

$$H(v) \leq \text{const} \cdot H(u) \quad \text{for all } v \in B_{d_\Omega(u)/2}^\Omega(x)$$

(see Lemma A.1 and Remark A.3). Moreover, since both functions  $H$  vanish on  $\partial\Omega$  near  $u_n$ , the weak Beurling estimate (see Lemma A.4) provides us a uniform bound

$$H(u_k) \leq \text{const} \cdot (d_\Omega(u_k)/d_\Omega(u))^{\beta_0} H(u), \quad k \geq m,$$

where  $\beta_0 > 0$  is some absolute constant. In particular, the values  $H(u_k)$  decay exponentially when  $k$  grows (as  $d_\Omega(u_k)$  do), and we easily finish the proof repeating the argument (3.7) given above:

$$\begin{aligned} \mathbb{P}[\text{RW}_\Omega(a; b) \cap L''_u \neq \emptyset] &\leq \sum_{k=m}^{n-1} \frac{Z_\Omega(u_k; a)Z_\Omega(u_k; b)}{Z_\Omega(a; b)} \\ &\leq \text{const} \cdot \frac{Z_\Omega(u; a)Z_\Omega(u; b)}{Z_\Omega(a; b)} \asymp \mathbb{P}_\Omega^{a,b}[B_\Omega(u)] \end{aligned}$$

(the terms decay exponentially as  $k$  grows, so the final bound do *not* depend on  $n$ ).  $\square$

**Proposition 3.3.** *Let  $\Omega$  be a simply connected discrete domain,  $a, b \in \partial\Omega$ ,  $u \in \text{Int } \Omega$ , and  $\sigma > 0$  be such that both  $\omega_\Omega(u; [ab]_\Omega)$ ,  $\omega_\Omega(u; [ba]_\Omega) \geq \sigma$ . Then, the uniform estimate*

$$\mathbb{P}_\Omega^{a,b}[B_\Omega(u)] \geq \text{const}(\sigma) \quad (3.8)$$

holds true, with some  $\text{const}(\sigma) > 0$  independent of  $\Omega, a, b, u$ .

*Proof.* For simplicity, let us re-scale the underlying graph  $\Gamma$  so that  $d_\Omega(u) = 1$ . The weak Beurling estimate (Lemma A.4) and our assumption on the harmonic measures of the arcs  $[ab]_\Omega$  and  $[ba]_\Omega$  seen from  $u$  imply that there exists big  $R = R(\sigma)$  (below we usually omit  $\sigma$ , assuming that it is fixed throughout the proof) such that

$$u \text{ is connected to both } [ab]_\Omega \text{ and } [ba]_\Omega \text{ inside } B_R^\Omega(u).$$

Indeed, if  $u$  is not connected to, say,  $[ba]_\Omega$ , and  $R$  is big enough, then the harmonic measure  $\omega_\Omega(u; [ba]_\Omega)$  should be too small (recall that we set  $d_\Omega(u) = 1$ ).

Due to Remark 2.2(ii), without loss of generality, we may assume that  $r_v \leq \delta$  for all  $v \in B_{2R}^\Omega(u)$ , where  $\delta$  is small enough (otherwise, the number of vertices in  $B_R^\Omega(u)$  is uniformly bounded, and (3.8) is guaranteed by the fact that any random walk path connecting  $a$  and  $b$  in  $\Omega$  should visit  $B_R^\Omega(u)$  due to topological reasons).

Moreover, it also follows from the weak Beurling estimate that there exist small  $r = r(\sigma) > 0$  such that

$$\begin{aligned} u \text{ remains connected to } [ba]_\Omega \text{ in } \Omega_{2R, 2r} &:= B_{2R}^\Omega(u) \setminus \bigcup_{x \in [ab]_\Omega} B_{2r}^\Omega(x) \\ &\text{(and vice versa with } [ba]_\Omega \text{ and } [ab]_\Omega \text{ interchanged)}. \end{aligned}$$

Indeed, if  $\bigcup_{x \in [ab]_\Omega} B_{2r}^\Omega(x)$  separates  $u$  from  $[ba]_\Omega$  in  $B_{2R}^\Omega(u)$ , then there exists a nearest-neighbor path  $\gamma$  separating  $u$  and  $[ba]_\Omega$  in  $\Omega$  such that  $\text{diam } \gamma \leq 5r$  and  $\text{dist}(u; \gamma) \geq \frac{1}{2}$ , which forces  $\omega_\Omega(u; [ba]_\Omega)$  to be too small (see the second part of Lemma A.4).

Now let  $L$  denote some discrete path running from  $u$  to  $[ba]_\Omega$  in  $\Omega_{2R, 2r}$ . We define a sequence of vertices  $u = u_0, u_1, \dots \in L$  inductively by the following rule:  $u_k$  is the first vertex on  $L$  which does not belong to  $\bigcup_{s < k} \text{Int } B_\Omega(u_s)$ . Let  $u_m$  be the first of those  $u_k$  satisfying  $d_\Omega(u_m) < \varepsilon_0 r$ . In particular, the length of the shortest discrete path  $L_{u_m}$  connecting  $u_m$  to  $\partial\Omega$  is less than  $r$ , see Remark 2.1. Hence, this path necessarily ends at the boundary arc  $(ba)_\Omega$  as  $u_m \notin \bigcup_{x \in [ab]_\Omega} B_{2r}^\Omega(x)$ .

Similarly to the proof of Lemma 3.2, we denote the parts of  $L$  from  $u$  to  $u_m$  and from  $u_m$  to  $\partial\Omega$  by  $L'$  and  $L''$ , respectively. Note that  $m$  is uniformly bounded. Indeed, by Remark 2.2(i) and our construction of  $u_k$ , one has  $|u_k - u_s| \geq \varepsilon_0 \cdot \frac{1}{4} d_\Omega(u_s) \geq \frac{1}{4} \varepsilon_0^2 r$  for all  $0 \leq s < k \leq m$ . As all  $u_k$  lie inside  $B_{2R}^\Omega(u)$ , one has  $m \leq \text{const}$ . Applying Harnack's principle and Proposition 3.1 similarly to (3.7), one arrives at the uniform estimate

$$\mathbb{P}[\text{RW}_\Omega(a; b) \cap L' \neq \emptyset] \leq \sum_{k=0}^m \mathbb{P}_\Omega^{a,b}[B_\Omega(u_k)] \leq \text{const} \cdot \mathbb{P}_\Omega^{a,b}[B_\Omega(u)].$$

Finally, we apply Lemma 3.2 for the vertex  $u_m$  (recall that  $d = \text{Length}(L_{u_m}) \leq r$  and  $u_m \notin B_{2r}^\Omega(a) \cup B_{2r}^\Omega(b)$  due to our choice of  $r$ ). It gives

$$\mathbb{P}[\text{RW}_\Omega(a; b) \cap L_{u_m} \neq \emptyset] \leq \text{const} \cdot \mathbb{P}_\Omega^{a,b}[B_\Omega(u_m)] \asymp \mathbb{P}_\Omega^{a,b}[B_\Omega(u)].$$

Summarizing, for a nearest-neighbor path  $L' \cup L_{u_m}$  running from  $u$  to  $(ba)_\Omega$ , we have

$$\mathbb{P}[\text{RW}_\Omega(a; b) \cap (L' \cup L_{u_m}) \neq \emptyset] \leq \text{const} \cdot \mathbb{P}_\Omega^{a,b}[B_\Omega(u)].$$

Clearly, one can repeat the same arguments for the other boundary arc  $(ab)_\Omega$ . We finish the proof by saying that, due to topological reasons,  $\text{RW}_\Omega(a; b)$  should cross at least one of those two paths (joining  $u$  with  $(ba)_\Omega$  and  $(ab)_\Omega$ , respectively).  $\square$

The last ingredient of the proof of Theorem 3.5 is the following simple

**Lemma 3.4.** *There exist a constant  $\sigma_0 > 0$  such that, for any simply connected discrete domain  $\Omega$  and three boundary points  $a, b, c \in \partial\Omega$  listed counterclockwise, one can find an inner point  $u \in \text{Int } \Omega$  so that all*

$$\omega_\Omega(u; [ab]_\Omega), \omega_\Omega(u; [bc]_\Omega), \omega_\Omega(u; [ca]_\Omega) \geq \sigma_0. \quad (3.9)$$

*Proof.* Recall that “no flat angles” assumption (see Sect. 2.1) guarantees that all degrees of faces of  $\Gamma$  are uniformly bounded. Let

$$\text{Int } \Omega_{[ab]} := \{u \in \text{Int } \Omega : \omega_\Omega(u; [ab]_\Omega) \geq \sigma\}.$$

If  $\sigma$  is chosen small enough (independently of  $(\Omega; a, b, c)$ ), then  $\Omega_{[ab]}$  is connected, containing all the vertices of faces touching  $[ab]_\Omega$  (in particular,  $\partial\Omega_{[ab]} \supset [ab]_\Omega$ ). Moreover, due to the maximum principle,  $\Omega_{[ab]}$  is simply connected. Let

$$L_{[ba]} := \partial\Omega_{[ab]} \setminus [ab]_\Omega,$$

$b^+ \in L_{[ba]}$  denote the next vertex on  $\partial\Omega_{[ab]}$  after  $b$ , and  $a^- \in L_{[ba]}$  denote the vertex just before  $a$ , if going along  $\partial\Omega_{[ab]}$  counterclockwise (in other words,  $b^+$  and  $a^-$  are the endpoints of  $L_{[ba]}$  viewed as a boundary arc of  $\Omega_{[ab]}$ ). For  $x \in L_{[ba]}$ , let  $x_{\text{int}} \in \text{Int } \Omega_{[ab]}$  denote the corresponding interior vertex. Then,

$$\begin{aligned} \omega_\Omega(x_{\text{int}}; [bc]_\Omega) + \omega_\Omega(x_{\text{int}}; [ca]_\Omega) &\geq \text{const} \cdot \omega_\Omega(x; [bc]_\Omega \cup [ca]_\Omega) \\ &\geq \text{const} \cdot (1 - \omega_\Omega(x; [ab]_\Omega)) \geq \text{const} \cdot (1 - \sigma) \end{aligned}$$

for all  $x \in L_{[ba]}$ . Moreover, for any two consecutive vertices  $x, x' \in L_{[ba]}$ , the corresponding vertices  $x_{\text{int}}$  and  $x'_{\text{int}}$  belong to the same face of  $\Gamma$ , so

$$\omega_\Omega(x'_{\text{int}}; [bc]_\Omega) \asymp \omega_\Omega(x_{\text{int}}; [bc]_\Omega) \quad \text{and} \quad \omega_\Omega(x'_{\text{int}}; [ca]_\Omega) \asymp \omega_\Omega(x_{\text{int}}; [ca]_\Omega).$$

On the other hand,  $\omega_\Omega(b_{\text{int}}^+; [bc]_\Omega) \geq \text{const}$  and  $\omega_\Omega(a_{\text{int}}^-; [ca]_\Omega) \geq \text{const}$  due to the same argument (e.g.,  $b_{\text{int}}^+$  shares a face with  $b_{\text{int}}$ ). Therefore, observing  $L_{[ba]}$  step by step, one can find  $x \in L_{[ba]}$  such that both  $\omega_\Omega(x_{\text{int}}; [bc]_\Omega)$  and  $\omega_\Omega(x_{\text{int}}; [ca]_\Omega)$  are bounded below by some constant (and  $\omega_\Omega(x_{\text{int}}; [ab]_\Omega) \geq \sigma$  as  $x_{\text{int}} \in \text{Int } \Omega_{[ab]}$ ).  $\square$

**Theorem 3.5.** *Let  $\Omega$  be a simply connected discrete domain and  $a, b, c \in \partial\Omega$  be listed counterclockwise. Then, the following double-sided estimate is fulfilled:*

$$Z_\Omega(a; [bc]_\Omega) \asymp \left[ \frac{Z_\Omega(a; b) Z_\Omega(a; c)}{Z_\Omega(b; c)} \right]^{\frac{1}{2}}, \quad (3.10)$$

with some absolute (i.e., independent of  $\Omega, a, b, c$ ) constants.

*Proof.* Due to Lemma 3.4, one can find an inner point  $u \in \text{Int } \Omega$  such that (3.9) holds true, where  $\sigma_0$  is independent of  $\Omega, a, b, c$ . Since, for any  $x \in [bc]_\Omega$ , one has

$$\omega_\Omega(u; [ax]_\Omega) \geq \omega_\Omega(u; [ab]_\Omega) \geq \sigma_0 \quad \text{and} \quad \omega_\Omega(u; [xa]_\Omega) \geq \omega_\Omega(u; [ca]_\Omega) \geq \sigma_0,$$

Propositions 3.1 and 3.3 imply

$$\begin{aligned} Z_\Omega(a; [bc]_\Omega) &= \sum_{x \in [bc]_\Omega} Z_\Omega(a; x) \asymp \sum_{x \in [bc]_\Omega} Z_\Omega(u; a) Z_\Omega(u; x) \\ &= Z_\Omega(u; a) Z_\Omega(u; [bc]_\Omega) \asymp Z_\Omega(u; a), \end{aligned}$$

where we have used  $Z_\Omega(u; [bc]_\Omega) \asymp \omega_\Omega(u; [bc]_\Omega) \asymp 1$ , and

$$\left[ \frac{Z_\Omega(a; b) Z_\Omega(a; c)}{Z_\Omega(b; c)} \right]^{\frac{1}{2}} \asymp \left[ \frac{Z_\Omega(u; a) Z_\Omega(u; b) \cdot Z_\Omega(u; a) Z_\Omega(u; c)}{Z_\Omega(u; b) Z_\Omega(u; c)} \right]^{\frac{1}{2}} = Z_\Omega(u; a)$$

as well.  $\square$

#### 4. DISCRETE CROSS-RATIOS

The main purpose of this Section is to obtain a uniform double-sided estimate (4.4) relating discrete analogues of two conformal invariants defined for a simply connected discrete domain  $\Omega$  with four marked boundary points  $a, b, c, d$ : discrete cross-ratio  $Y_\Omega(a, b; c, d)$  (see Definition 4.3) and the total partition function  $Z_\Omega([ab]_\Omega; [cd]_\Omega)$  of random walks connecting two opposite boundary arcs. Note that the cross-ratio  $Y_\Omega$  changes to its reciprocal when replacing boundary arcs  $[ab]_\Omega$  and  $[cd]_\Omega$  by “dual” ones  $[bc]_\Omega$  and  $[da]_\Omega$ , while the corresponding change of  $Z_\Omega$  is more sophisticated (see (4.4)).

**4.1. Monotonicity of  $Z_\Omega(x; a)/Z_\Omega(x; b)$  on the boundary and definition of discrete cross-ratios.** Let two points  $a, b$  (or, more generally, two disjoint arcs  $A = [a_1 a_2]_\Omega$  and  $B = [b_1 b_2]_\Omega$ ) on the boundary of a simply connected discrete domain  $\Omega$  be fixed. Then, one can use the ratio  $Z_\Omega(x; a)/Z_\Omega(x; b)$  in order to “track” the position of  $x$  with respect to  $a, b$ . Being considered on  $\partial\Omega$ , this ratio has a monotonicity property (see Lemma 4.1 below), which allows one to use it as a “parametrization” of  $\partial\Omega$  between  $A$  and  $B$ . Namely, for  $x \in \partial\Omega$ , denote

$$R_\Omega(x; A, B) := \frac{Z_\Omega(x; A)}{Z_\Omega(x; B)}.$$

**Lemma 4.1.** *Let  $\Omega$  be a simply connected discrete domain and  $A = [a_1 a_2]_\Omega$ ,  $B = [b_1 b_2]_\Omega$  denote two disjoint boundary arcs of  $\Omega$ . Then,  $R_\Omega(\cdot; A, B)$  decreases along the boundary arc  $[a_2 b_1]_\Omega$  and increases along the boundary arc  $[b_2 a_1]_\Omega$ .*

**Remark 4.2.** In particular, if  $A = \{a\}$  and  $B = \{b\}$  are just single boundary points, then  $R_\Omega(\cdot; a, b)$  attains its maximal and minimal values on  $\partial\Omega$  at  $a$  and  $b$ , respectively, being monotone on both boundary arcs  $[ab]_\Omega$  and  $[ba]_\Omega$ .

*Proof.* Similarly to the proof of Remark 2.6(i), for any given  $t > 0$ , we define a discrete harmonic (in  $\Omega$ ) function

$$H_t(u) := \begin{cases} Z_\Omega(u; A) - t Z_\Omega(u; B), & u \in \text{Int } \Omega, \\ \mu_u^{-1}(\mathbb{1}_A(u) - t \mathbb{1}_B(u)), & u \in \partial\Omega. \end{cases}$$

Note that, for any  $x \in \partial\Omega$ , one has

$$Z_\Omega(x; A) - t Z_\Omega(x; B) = H_t(x) + \varpi_{xx_{\text{int}}} H_t(x_{\text{int}}).$$

For a given boundary point  $x \in (a_2 b_1]_\Omega$ , let  $t_x > 0$  be chosen so that  $H_{t_x}(x_{\text{int}}) = 0$  (if  $x \in (a_2 b_1)_\Omega$ , this means  $R_\Omega(x; A, B) = t_x$  as  $H_{t_x}(x) = 0$ , while  $R_\Omega(b_1; A, B) < t_{b_1}$ ).

The function  $H_{t_x}$  is discrete harmonic in  $\Omega$ , vanishes on  $\partial\Omega \setminus (A \cup B)$ , is strictly positive on  $A$  and strictly negative on  $B$ . Therefore, there exists a nearest-neighbor path  $\gamma_{xA}$  running from  $x_{\text{int}}$  to  $A$  such that  $H_{t_x} \geq 0$  along  $\gamma_{xA}$ . Due to the maximum

principle, this implies  $H_{t_x}(y_{\text{int}}) \geq 0$  for all intermediate boundary points  $y \in [a_2x]_\Omega$ . In other words,

$$Z_\Omega(y; A) - t_x Z_\Omega(y; B) = \mu_{a_2}^{-1} \mathbb{1}[y=a_2] + \varpi_{y y_{\text{int}}} H_{t_x}(y_{\text{int}}) \geq 0 \text{ for all } y \in [a_2x]_\Omega.$$

Thus,  $R_\Omega(y; A, B) \geq t_x \geq R_\Omega(x; A, B)$  for all  $y \in [a_2x]_\Omega$  which means that  $R_\Omega(\cdot; A, B)$  decreases along  $[a_2b_1]_\Omega$ . The proof for the other boundary arc  $[b_2a_1]_\Omega$  is similar.  $\square$

**Definition 4.3.** Let  $\Omega$  be a simply-connected discrete domain and boundary points  $a, b, c, d \in \partial\Omega$  be listed counterclockwise. We define their **discrete cross-ratios** by

$$\mathbf{X}_\Omega(a, b; c, d) := \left[ \frac{Z_\Omega(a; c) \cdot Z_\Omega(b; d)}{Z_\Omega(a; b) \cdot Z_\Omega(c; d)} \right]^{\frac{1}{2}}; \quad \mathbf{Y}_\Omega(a, b; c, d) := \left[ \frac{Z_\Omega(a; d) \cdot Z_\Omega(b; c)}{Z_\Omega(a; b) \cdot Z_\Omega(c; d)} \right]^{\frac{1}{2}}.$$

**Remark 4.4.** Since  $a, b, c, d$  are listed counterclockwise, Lemma 4.1 implies

$$\mathbf{X}_\Omega(a, b; c, d) = \left[ \frac{R_\Omega(a; c, b)}{R_\Omega(d; c, b)} \right]^{\frac{1}{2}} \leq 1 \quad \text{and} \quad \frac{\mathbf{X}_\Omega(a, b; c, d)}{\mathbf{Y}_\Omega(a, b; c, d)} = \left[ \frac{R_\Omega(a; c, d)}{R_\Omega(b; c, d)} \right]^{\frac{1}{2}} \leq 1.$$

We include the exponent  $\frac{1}{2}$  in Definition 4.3 by two (clearly related) reasons: first, it simplifies several double-sided estimates given below, and, second, it makes the notation closer to the standard continuous setup. Indeed, the continuous analogue of the partition function  $Z_\Omega(a; b)$  for the upper half-plane  $\mathbb{H}$  (up to a multiplicative constant) is given by  $(b-a)^{-2}$ , so the quantities  $\mathbf{X}_\Omega$  and  $\mathbf{Y}_\Omega$  introduced above are “discrete versions” (in  $\Omega$ ) of the usual cross-ratios

$$x_{\mathbb{H}}(a, b; c, d) := \frac{(b-a)(d-c)}{(c-a)(d-b)} \quad \text{and} \quad y_{\mathbb{H}}(a, b; c, d) := \frac{(b-a)(d-c)}{(d-a)(c-b)}.$$

In the continuous setup, the following is fulfilled:  $(x_{\mathbb{H}}(a, b; c, d))^{-1} \equiv 1 + (y_{\mathbb{H}}(a, b; c, d))^{-1}$ . One clearly cannot hope that the same *identity* remains valid on the discrete level for *all*  $\Omega$ 's (even, say, if  $\Gamma$  is the standard square grid). Nevertheless, below we prove that the similar *uniform double-sided estimate* holds true for the discrete cross-ratios, with constants, in general, depending on parameters fixed in assumptions (a)–(d), (S), (T), but *not* on the configuration  $(\Omega; a, b, c, d)$  or the underlying graph  $\Gamma$  structure.

**Proposition 4.5.** Let  $\Omega$  be a simply connected discrete domain and  $a, b, c, d \in \partial\Omega$  be listed counterclockwise. Then the following double-sided estimate holds true:

$$(\mathbf{X}_\Omega(a, b; c, d))^{-1} \asymp 1 + (\mathbf{Y}_\Omega(a, b; c, d))^{-1}, \quad (4.1)$$

with some absolute (i.e., independent of  $\Omega, a, b, c, d$ ) constants.

*Proof.* We apply factorization (3.10) to both sides of the trivial double-sided estimate

$$Z_\Omega(a; [bd]_\Omega) \asymp Z_\Omega(a; [bc]_\Omega) + Z_\Omega(a; [cd]_\Omega),$$

which is almost an identity besides the term  $Z_\Omega(a; c)$  counted once in the left-hand side and twice in the right-hand side. Dividing by  $[Z_\Omega(a; b)Z_\Omega(a; c)Z_\Omega(a; d)]^{1/2}$ , one obtains the following double-sided estimate:

$$\left[ \frac{1}{Z_\Omega(a; c)Z_\Omega(b; d)} \right]^{\frac{1}{2}} \asymp \left[ \frac{1}{Z_\Omega(a; d)Z_\Omega(b; c)} \right]^{\frac{1}{2}} + \left[ \frac{1}{Z_\Omega(a; b)Z_\Omega(c; d)} \right]^{\frac{1}{2}}, \quad (4.2)$$

which is equivalent to (4.1).  $\square$



**Remark 4.6.** It immediately follows from (4.1) that  $X_\Omega(a, b; c, d) \asymp Y_\Omega(a, b; c, d)$ , if  $Y_\Omega \leq \text{const}$  (which means that arcs  $[ab]_\Omega$  and  $[cd]_\Omega$  are “not too close” in  $\Omega$ ). Moreover, the next Proposition shows that, in this case,  $Z_\Omega([ab]_\Omega; [cd]_\Omega) \asymp Y_\Omega(a, b; c, d)$  as well, since  $Z_\Omega$  is always squeezed (up to multiplicative constants) by  $X_\Omega$  and  $Y_\Omega$ .

**Proposition 4.7.** *Let  $\Omega$  be a simply connected discrete domain and  $a, b, c, d \in \partial\Omega$  be listed in the counterclockwise order. Then, the following estimates are fulfilled:*

$$\text{const} \cdot X_\Omega(a, b; c, d) \leq Z_\Omega([ab]_\Omega; [cd]_\Omega) \leq \text{const} \cdot Y_\Omega(a, b; c, d), \quad (4.3)$$

with some absolute (i.e., independent of  $\Omega, a, b, c, d$ ) constants.

*Proof.* Due to Theorem 3.5, one has

$$Z_\Omega([ab]_\Omega; [cd]_\Omega) = \sum_{x \in [ab]_\Omega} Z_\Omega(x; [cd]_\Omega) \asymp \frac{1}{(Z_\Omega(c; d))^{\frac{1}{2}}} \sum_{x \in [ab]_\Omega} (Z_\Omega(x; c))^{\frac{1}{2}} (Z_\Omega(x; d))^{\frac{1}{2}}.$$

It follows from Lemma 4.1 that, for any  $x \in [ab]_\Omega$ ,

$$(Z_\Omega(x; c))^{\frac{1}{2}} (Z_\Omega(x; d))^{\frac{1}{2}} = \frac{(Z_\Omega(x; c))^{\frac{1}{2}}}{(Z_\Omega(x; d))^{\frac{1}{2}}} \cdot Z_\Omega(x; d) \geq \frac{(Z_\Omega(a; c))^{\frac{1}{2}}}{(Z_\Omega(a; d))^{\frac{1}{2}}} \cdot Z_\Omega(x; d).$$

Therefore, summing and applying Theorem 3.5 once more, one obtains

$$\begin{aligned} Z_\Omega([ab]_\Omega; [cd]_\Omega) &\geq \text{const} \cdot \frac{(Z_\Omega(a; c))^{\frac{1}{2}}}{(Z_\Omega(c; d))^{\frac{1}{2}} (Z_\Omega(a; d))^{\frac{1}{2}}} \cdot Z_\Omega([ab]_\Omega; d) \\ &\asymp \frac{(Z_\Omega(a; c))^{\frac{1}{2}} (Z_\Omega(b; d))^{\frac{1}{2}}}{(Z_\Omega(c; d))^{\frac{1}{2}} (Z_\Omega(a; b))^{\frac{1}{2}}} = X_\Omega(a, b; c, d). \end{aligned}$$

On the other hand, Cauchy’s inequality (and Theorem 3.5 again) gives

$$\begin{aligned} (Z_\Omega([ab]_\Omega; [cd]_\Omega))^2 &\leq \text{const} \cdot \frac{Z_\Omega([ab]_\Omega; c) Z_\Omega([ab]_\Omega; d)}{Z_\Omega(c; d)} \\ &\asymp \frac{(Z_\Omega(a; c) Z_\Omega(b; c) Z_\Omega(a; d) Z_\Omega(b; d))^{\frac{1}{2}}}{Z_\Omega(c; d) Z_\Omega(a; b)} \\ &= X_\Omega(a, b; c, d) Y_\Omega(a, b; c, d) \leq (Y_\Omega(a, b; c, d))^2. \quad \square \end{aligned}$$

**Theorem 4.8.** *Let  $\Omega$  be a simply connected discrete domain and  $a, b, c, d \in \partial\Omega$  be listed counterclockwise. Then, the following double-sided estimate holds true:*

$$Z_\Omega([ab]_\Omega; [cd]_\Omega) \asymp \log(1 + Y_\Omega(a, b; c, d)), \quad (4.4)$$

with some absolute (i.e., independent of  $\Omega, a, b, c, d$ ) constants.

*Proof.* Denote  $Y_\Omega := Y_\Omega(a, b; c, d)$ ,  $X_\Omega := X_\Omega(a, b; c, d)$ , and let a constant  $M$  be chosen big enough (independently of  $(\Omega; a, b, c, d)$ ). If  $Y_\Omega \leq M$ , Propositions 4.5, 4.7 imply

$$\begin{aligned} Z_\Omega([ab]; [cd]) &\geq \text{const} \cdot X_\Omega \asymp (1 + Y_\Omega)^{-1} Y_\Omega \geq (1 + M)^{-1} \cdot \log(1 + Y_\Omega), \\ Z_\Omega([ab]; [cd]) &\leq \text{const} \cdot Y_\Omega \leq \text{const} \cdot M [\log(1 + M)]^{-1} \cdot \log(1 + Y_\Omega) \end{aligned} \quad (4.5)$$

(with constants independent of  $M$ ). Thus, without loss of generality, we can assume that  $Y_\Omega \geq M$  (i.e.,  $[ab]_\Omega$  and  $[cd]_\Omega$  are “very close” to each other in  $\Omega$ ). Let

$$R_\Omega(x) := R_\Omega(x; c, d) = \frac{Z_\Omega(x; c)}{Z_\Omega(x; d)}, \quad x \in [ab]_\Omega.$$

Due to Lemma 4.1,  $R_\Omega$  increases on  $[ab]_\Omega$ . Moreover, it follows from Proposition 4.5 (or directly from (4.2)) that

$$\left[ \frac{R_\Omega(b)}{R_\Omega(a)} \right]^{\frac{1}{2}} = \left[ \frac{Z_\Omega(b; c)Z_\Omega(a; d)}{Z_\Omega(b; d)Z_\Omega(a; c)} \right]^{\frac{1}{2}} \asymp 1 + \left[ \frac{Z_\Omega(b; c)Z_\Omega(a; d)}{Z_\Omega(a; b)Z_\Omega(c; d)} \right]^{\frac{1}{2}} = 1 + Y_\Omega \asymp Y_\Omega.$$

As any two consecutive boundary vertices  $x, x' \in [ab]_\Omega$  belong to the same face of  $\Gamma$ , one has  $Z_\Omega(x; c) \asymp Z_\Omega(x'; c)$ ,  $Z_\Omega(x; d) \asymp Z_\Omega(x'; d)$  and

$$1 \leq \frac{R_\Omega(x')}{R_\Omega(x)} \leq \text{const}.$$

Therefore, provided that  $Y_\Omega \geq M$  is big enough, one can find a number  $n \asymp \log Y_\Omega$  and a sequence of boundary points  $a = a_0, a_1, \dots, a_n = b$  such that

$$4 \leq \frac{R_\Omega(a_{k+1})}{R_\Omega(a_k)} \leq \text{const}$$

for all  $k = 0, \dots, n-1$ . This can be easily rewritten as

$$\text{const} \leq \left[ \frac{R_\Omega(a_k)}{R_\Omega(a_{k+1})} \right]^{\frac{1}{2}} = X_\Omega(a_k, a_{k+1}; c, d) \leq \frac{1}{2},$$

or, due to Proposition 4.5, as  $Y_\Omega(a_k, a_{k+1}; c, d) \asymp 1$ . Hence, if the constant  $M$  was chosen big enough, the estimate (4.5) implies

$$Z_\Omega([a_k a_{k+1}]_\Omega; [cd]_\Omega) \asymp 1$$

for all  $k = 0, \dots, n-1$ . This easily gives

$$Z_\Omega([ab]_\Omega; [cd]_\Omega) \asymp \sum_{k=0}^{n-1} Z_\Omega([a_k a_{k+1}]_\Omega; [cd]_\Omega) \asymp n \asymp \log Y_\Omega. \quad (4.6)$$

Combining estimates (4.5) for  $Y_\Omega \leq M$  and (4.6) for  $Y_\Omega \geq M$ , one arrives at (4.4).  $\square$

## 5. SURGERY TECHNIQUE

The main purpose of this Section is to illustrate how tools developed above can be used to construct cross-cuts of a simply connected discrete domain  $\Omega$  having some nice “separation” properties, without any reference to the actual geometry of  $\Omega$ . The main result is Theorem 5.1 which claims the existence of those “separators”. In Proposition 5.2, we also give some simple monotonicity properties of such cross-cuts.

More precisely, let  $A = [a_1 a_2]_\Omega$  and  $B = [b_1 b_2]_\Omega$  be two disjoint boundary arcs of a simply connected  $\Omega$ . We are interested in the following question: whether it is possible to cut  $\Omega$  along some cross-cut  $L$  into two simply connected parts  $\Omega_A, \Omega_B$ , one containing  $A$  and the other containing  $B$ , so that

$$Z_\Omega(A; B) \asymp Z_{\Omega_A}(A; L) Z_{\Omega_B}(L; B) ? \quad (5.1)$$

Moreover, we are interested not only in a *single* cross-cut  $L$ , but rather a *family*  $L_k = L_A^B[k]$  such that, in addition to factorization (5.1), one has

$$Z_{\Omega_A}(A; L_k) / Z_{\Omega_B}(L_k; B) \asymp k. \quad (5.2)$$

Note that both  $Z_{\Omega_A}(A; L_k), Z_{\Omega_B}(L_k; B) \geq Z_{\Omega}(A; B)$ . Thus, if  $Z_{\Omega}(A; B) \gg 1$ , then (5.1) certainly fails. Due to the same reason, one cannot hope for (5.2), if  $k \ll Z_{\Omega}(A; B)$  or  $k \gg (Z_{\Omega}(A; B))^{-1}$ . However, being motivated by the continuous setup, one certainly hopes for the positive answer in all other situations and, indeed, Theorem 5.1 claims the existence of a “separator”  $L_A^B[k]$  and provides a natural construction of this slit for any given  $\Omega, A, B$  and  $k$ .

Namely, let discrete domains  $\Omega_A^B[k]$  and  $\Omega_B^A(k^{-1})$  be defined by

$$\begin{aligned} \text{Int } \Omega_A^B[k] &:= \left\{ u \in \text{Int } \Omega : \frac{Z_{\Omega}(u; A)}{Z_{\Omega}(u; B)} \geq k \right\}, \\ \text{Int } \Omega_B^A(k^{-1}) &:= \left\{ u \in \text{Int } \Omega : \frac{Z_{\Omega}(u; B)}{Z_{\Omega}(u; A)} > k^{-1} \right\} \end{aligned}$$

(we use square and round brackets to abbreviate  $\geq$  and  $>$  inequalities, respectively). Below we always work with  $k$ 's which are not extremely big or extremely small, so that  $\text{Int } \Omega_A^B[k]$  contains all vertices of faces touching  $A$ , while  $\text{Int } \Omega_B^A(k^{-1})$  contains all vertices near  $B$ . Then, both  $\Omega_A^B[k]$  and  $\Omega_B^A(k^{-1})$  are simply connected due to the maximum principle applied to the discrete harmonic function  $Z_{\Omega}(\cdot; A) - kZ_{\Omega}(\cdot; B)$ . Further, we denote the set of edges

$$L_A^B[k] = L_B^A(k^{-1}) := \{(u_A u_B) \in E_{\text{int}}^{\Omega} : u_A \in \text{Int } \Omega_A^B[k], u_B \in \text{Int } \Omega_B^A(k^{-1})\}$$

(see Fig. 2A). According to our conventions concerning the boundary of a discrete domain, this set can be interpreted as a part of  $\partial\Omega_A^B[k]$ , as well as a part of  $\partial\Omega_B^A(k^{-1})$ .

**Theorem 5.1.** *Let  $\Omega$  be a simply connected discrete domain,  $A, B \subset \partial\Omega$  be two disjoint boundary arcs,  $Z := Z_{\Omega}(A; B)$ , and  $k > 0$  be chosen so that both  $\Omega_A := \Omega_A^B[k]$  and  $\Omega_B := \Omega_B^A(k^{-1})$  are connected (i.e.,  $\Omega_A$  contains all inner vertices around  $A$  while  $\Omega_B$  contains all inner vertices around  $B$ ). Then,*

(i) *for any fixed (big) constant  $K \geq 1$ , the following is fulfilled:*

*if  $Z \leq K$  and  $K^{-1} \leq k \leq K$ , then the cross-cut  $L_k := L_A^B[k]$  satisfies both conditions (5.1), (5.2), with constants depending on  $K$  but independent of  $\Omega, A, B$  and  $k$ ;*

(ii) *there exists a (small) constant  $\kappa_0 > 0$  such that the following is fulfilled:*

*if  $Z \leq \kappa_0$  and  $\kappa_0^{-1}Z \leq k \leq \kappa_0 Z^{-1}$ , then the cross-cut  $L_k$  satisfies both conditions (5.1), (5.2) with some absolute constants. Moreover, in this case, both  $\Omega_A$  and  $\Omega_B$  are always connected.*

*Proof.* Since  $Z_{\Omega}(u_A; \cdot) \asymp Z_{\Omega}(u_B; \cdot)$ , it is clear that

$$\frac{Z_{\Omega}(u_A; A)}{Z_{\Omega}(u_A; B)} \asymp \frac{Z_{\Omega}(u_B; A)}{Z_{\Omega}(u_B; B)} \asymp k \quad \text{for all } u = (u_A u_B) \in L_k. \quad (5.3)$$

Let  $\partial\Omega_A \cap \partial\Omega = [y_A x_A]_{\Omega}$  and  $\partial\Omega_B \cap \partial\Omega = [x_B y_B]_{\Omega}$  (see Fig. 2A), and let

$$Z_A := Z_{\Omega}(A; [x_B y_B]_{\Omega}), \quad Z_B := Z_{\Omega}(B; [y_A x_A]_{\Omega}), \quad (5.4)$$

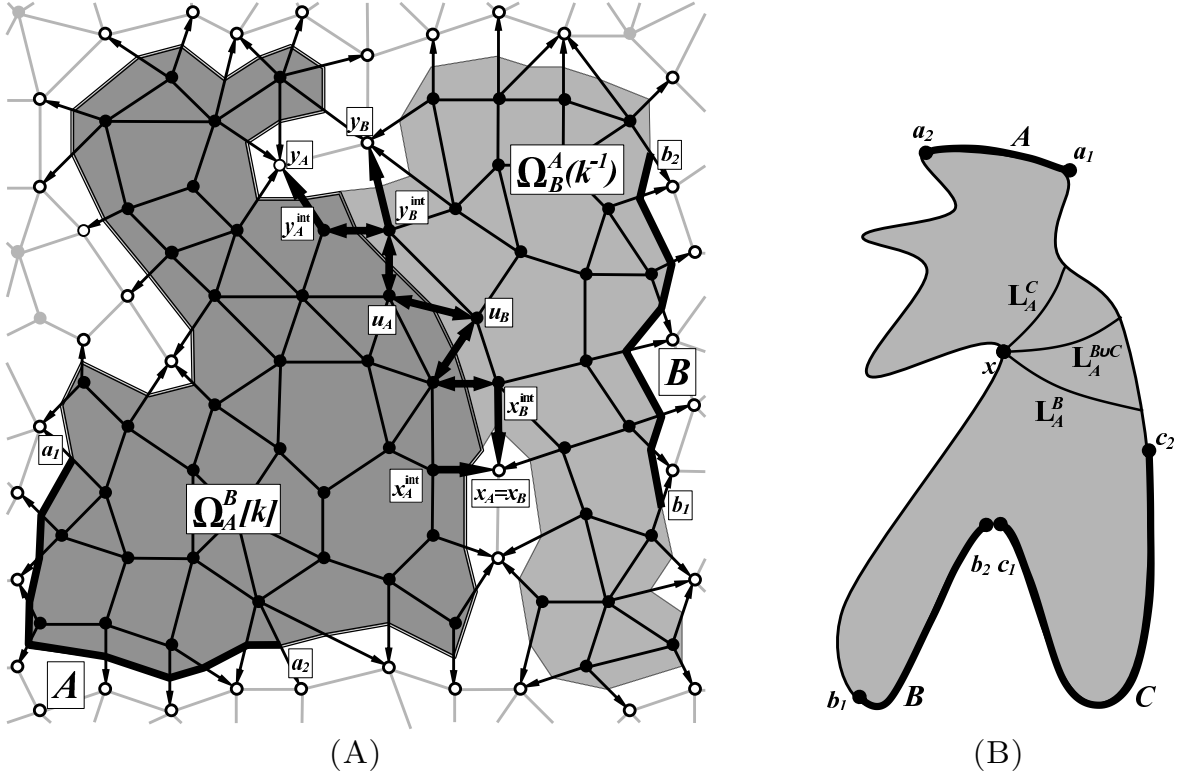


FIGURE 2. (A) A simply connected discrete domain split into two parts  $\Omega_A^B[k]$  and  $\Omega_B^A(k^{-1})$  according to the ratio of harmonic measures of two marked boundary arcs  $A = [a_1 a_2]$  and  $B = [b_1 b_2]$ . All edges  $(u_A u_B)$  that cross the slit  $L_A^B[k]$  are marked, as well as four boundary edges  $(x_A^{\text{int}} x_A), (x_B^{\text{int}} x_B), (y_B^{\text{int}} y_B), (y_A^{\text{int}} y_A) \in \partial\Omega$  neighboring to  $L_A^B[k]$ . (B) Notation used in Proposition 5.2 and schematic drawing of the monotonicity property  $\Omega_A^C[x] \subset \Omega_A^{B \cup C}[x] \subset \Omega_A^B[x]$  for  $x \in (a_2 b_1)$ .

where these partition functions are considered *in the original domain*  $\Omega$ . Then,

$$\begin{aligned} Z_{\Omega_A}(A; L_k) &= \sum_{u \in L_k} Z_{\Omega_A}(A; u_B) \asymp \sum_{u \in L_k} Z_{\Omega_A}(A; u_B) Z_{\Omega}(u_B; \partial\Omega) \\ &= \sum_{u \in L_k} Z_{\Omega_A}(A; u_B) Z_{\Omega}(u_B; [x_B y_B]_{\Omega}) + \sum_{u \in L_k} Z_{\Omega_A}(A; u_B) Z_{\Omega}(u_B; [y_A x_A]_{\Omega}), \end{aligned}$$

since  $Z_{\Omega}(u_B; \partial\Omega) \asymp 1$  for any  $u_B \in \text{Int } \Omega$ . Note that the first term can be rewritten as

$$\sum_{u \in L_k} Z_{\Omega_A}(A; u_B) Z_{\Omega}(u_B; [x_B y_B]_{\Omega}) \asymp Z_{\Omega}(A; [x_B y_B]_{\Omega}) = Z_A.$$

Indeed, each random walk path running from  $A$  to  $[x_B y_B]_{\Omega}$  inside  $\Omega$  should pass through  $L_k$  by topological reasons, so, denoting by  $u$  the *first* crossing, one obtains the result. Similarly, the second term is comparable to the total partition functions of those random walks which start from  $A$ , cross  $L_k$  (possibly many times), and finish back at

$[y_A x_A]_\Omega$ . Denoting by  $v$  the *last* crossing of  $L_k$  and using (5.3), one obtains

$$\begin{aligned} \sum_{u \in L_k} Z_{\Omega_A}(A; u_B) Z_\Omega(u_B; [y_A x_A]_\Omega) &\asymp \sum_{v \in L_k} Z_\Omega(A; v_B) Z_{\Omega_A}(v_B; [y_A x_A]_\Omega) \\ &\asymp k \sum_{v \in L_k} Z_\Omega(B; v_B) Z_{\Omega_A}(v_B; [y_A x_A]_\Omega) \asymp k Z_B, \end{aligned}$$

since each random walk path running from  $B$  to  $[y_A x_A]_\Omega$  inside  $\Omega$  should cross  $L_k$ . Thus, we arrive at the double-sided estimates

$$Z_{\Omega_A}(A; L_k) \asymp Z_A + k Z_B,$$

and, similarly,  $Z_{\Omega_B}(L_k; B) \asymp k^{-1} Z_A + Z_B$ . Therefore, it is sufficient to prove that

$$Z_A/Z_B \asymp k \quad \text{and} \quad Z_A Z_B \asymp Z. \quad (5.5)$$

It directly follows from (5.3) that

$$\frac{Z_\Omega(x; A)}{Z_\Omega(x; B)} \asymp k \asymp \frac{Z_\Omega(y; A)}{Z_\Omega(y; B)} \quad (5.6)$$

(here and below we omit subscripts of  $x$  and  $y$ , all the claims hold true for both  $x = x_A, x_B$  and, similarly,  $y = y_A, y_B$ , since the values of  $Z_\Omega(x_A; \cdot)$  and  $Z_\Omega(x_B; \cdot)$  are uniformly comparable). Denote

$$\begin{aligned} Y_A &:= Y_\Omega(a_1, a_2; x, y), & Y_B &:= Y_\Omega(b_1, b_2; y, x), \\ X_A &:= X_\Omega(a_1, a_2; x, y), & X_B &:= X_\Omega(b_1, b_2; y, x), \end{aligned}$$

where all discrete cross-ratios are considered *in the original domain*  $\Omega$ . Using Theorem 3.5 and (5.6), it is easy to check that

$$\left[ \frac{Y_A X_A}{Y_B X_B} \right]^{\frac{1}{2}} \asymp \left[ \frac{Z_\Omega(x; A) Z_\Omega(y; A)}{Z_\Omega(x; B) Z_\Omega(y; B)} \right]^{\frac{1}{2}} \asymp k. \quad (5.7)$$

The rest of the proof is divided into three steps:

- first, we prove (5.5) assuming that both  $Z_A, Z_B$  are bounded above by some absolute constant (roughly speaking, this means that  $x$  and  $y$  are “not too close” to both  $A, B$ ), in some sense this is the most conceptual step which is based on discrete cross-ratios technique developed in Sect. 4;
- second, we use discrete cross-ratios technique once again to show that, indeed, one has  $Z_A, Z_B \leq \text{const}$ , if  $k \asymp 1$  (in particular, this implies (i));
- last, we analyze general case in (ii) by starting with  $k = 1$  and then increasing it until  $Z_A$  becomes  $\asymp 1$ , which, as we show, cannot happen until  $k_{\max} \asymp Z^{-1}$ .

**Step 1. The proof of (5.5) under assumption  $Z_A, Z_B \leq \text{const}$ .** In this case Theorem 4.8 guarantees that  $Y_A, Y_B \leq \text{const}$  as well, and Remark 4.6 says that

$$Z_A \asymp [Y_A X_A]^{1/2} \quad \text{and} \quad Z_B \asymp [Y_B X_B]^{1/2}.$$

Therefore, (5.7) immediately gives the first part of (5.5). Moreover, one has  $X_A \asymp Y_A$  and  $X_B \asymp Y_B$ , which is equivalent to say that

$$\frac{Z_\Omega(x; a_1)}{Z_\Omega(y; a_1)} \asymp \frac{Z_\Omega(x; a_2)}{Z_\Omega(y; a_2)} \quad \text{and} \quad \frac{Z_\Omega(x; b_1)}{Z_\Omega(y; b_1)} \asymp \frac{Z_\Omega(x; b_2)}{Z_\Omega(y; b_2)}. \quad (5.8)$$

In addition, Theorem 3.5 applied to (5.6) gives

$$\frac{Z_\Omega(x; a_1)Z_\Omega(x; a_2)}{Z_\Omega(y; a_1)Z_\Omega(y; a_2)} \asymp \frac{Z_\Omega(x; b_1)Z_\Omega(x; b_2)}{Z_\Omega(y; b_1)Z_\Omega(y; b_2)},$$

thus upgrading (5.8) to

$$\frac{Z_\Omega(x; a_1)}{Z_\Omega(y; a_1)} \asymp \frac{Z_\Omega(x; a_2)}{Z_\Omega(y; a_2)} \asymp \frac{Z_\Omega(x; b_1)}{Z_\Omega(y; b_1)} \asymp \frac{Z_\Omega(x; b_2)}{Z_\Omega(y; b_2)}. \quad (5.9)$$

As  $Z \leq \text{const}$ , we also have  $Z \asymp X_\Omega(a_1, a_2; b_1, b_2)$ . Rearranging factors, one obtains

$$\frac{Z_A Z_B}{Z} \asymp \frac{[Y_A X_A Y_B X_B]^{\frac{1}{2}}}{X_\Omega(a_1, a_2; b_1, b_2)} \asymp [R_1 R_2]^{\frac{1}{4}}, \quad R_j := \frac{Z_\Omega(a_j; x)Z_\Omega(x; b_j)Z_\Omega(b_j; y)Z_\Omega(y; a_j)}{(Z_\Omega(a_j; b_j)Z_\Omega(x; y))^2}.$$

Finally, it directly follows from (5.9) that  $Y_\Omega(a_j, x; b_j, y) \asymp 1$ . Due to Proposition 4.5, this also implies  $X_\Omega(a_j, x; b_j, y) \asymp 1$  and, similarly,  $X_\Omega(x, b_j; y, a_j) \asymp 1$ . Therefore,

$$R_j = [X_\Omega(a_j, x; b_j, y)X_\Omega(x, b_j; y, a_j)]^{-\frac{1}{2}} \asymp 1,$$

i.e.,  $Z_A Z_B \asymp Z$  (which is the second part of (5.5)) and we are done.

**Step 2. Proof of  $Z_A, Z_B \leq \text{const}$ , if  $k \asymp 1$ .** In this case, Proposition 4.5 and (5.7) give

$$Y_A^2(1+Y_A)^{-1} \asymp Y_A X_A \asymp Y_B X_B \asymp Y_B^2(1+Y_B)^{-1}.$$

Thus, if, say,  $Y_A \leq \text{const}$ , then  $Y_B \leq \text{const}$  as well, and  $Z_A, Z_B \leq \text{const}$  due to Theorem 4.8. Hence, without loss of generality, we may assume that *both*  $Y_A, Y_B$  are bounded away from zero, which is equivalent to say that both  $X_A, X_B \asymp 1$ , i.e.,

$$Z_\Omega(x; a_1)Z_\Omega(y; a_2) \asymp Z_\Omega(x; y)Z_\Omega(a_1; a_2),$$

$$Z_\Omega(x; b_2)Z_\Omega(y; b_1) \asymp Z_\Omega(x; y)Z_\Omega(b_1; b_2).$$

Using Theorem 3.5 and (5.6), we obtain

$$\begin{aligned} \frac{Z_\Omega(x; a_2)}{Z_\Omega(y; a_2)} &\asymp \frac{Z_\Omega(x; a_2)Z_\Omega(x; a_1)}{Z_\Omega(a_1; a_2)Z_\Omega(x; y)} \asymp \frac{(Z_\Omega(x; A))^2}{Z_\Omega(x; y)} \\ &\asymp \frac{(Z_\Omega(x; B))^2}{Z_\Omega(x; y)} \asymp \frac{Z_\Omega(x; b_1)Z_\Omega(x; b_2)}{Z_\Omega(b_1; b_2)Z_\Omega(x; y)} \asymp \frac{Z_\Omega(x; b_1)}{Z_\Omega(y; b_1)}. \end{aligned}$$

Due to Proposition 4.5 applied for a discrete quadrilateral  $(\Omega; a_2, x; y, b_1)$ , it gives

$$Z_\Omega(x; a_2)Z_\Omega(y; b_1) \asymp Z_\Omega(x; y)Z_\Omega(a_2; b_1) \asymp Z_\Omega(x; b_1)Z_\Omega(y; a_2).$$

Similarly, one has  $Z_\Omega(x; a_1)Z_\Omega(y; b_2) \asymp Z_\Omega(x; y)Z_\Omega(a_1; b_2) \asymp Z_\Omega(x; b_2)Z_\Omega(y; a_1)$ . Then, using  $X_A, X_B \asymp 1$  and rearranging factors, one arrives at

$$Y_A Y_B \asymp Y_A X_A Y_B X_B \asymp \frac{Z_\Omega(a_1; b_2)Z_\Omega(a_2; b_1)}{Z_\Omega(a_1; a_2)Z_\Omega(b_2; b_1)} = Y_\Omega(a_1, a_2; b_1, b_2).$$

As  $Z$  is bounded above, Theorem 4.8 ensures that  $Y_\Omega(a_1, a_2; b_1, b_2) \leq \text{const}$ . Taking into account  $Y_A, Y_B \geq \text{const}$ , we arrive at  $Y_A, Y_B \asymp 1$ , and so  $Z_A, Z_B \asymp 1$ .

**Step 3. Proof of the general case in (ii).** Let  $Z_A(k)$  and  $Z_B(k)$  be defined by (5.4) for a given  $k$ . Note that  $Z_A(k), Z_B(k)$  are piecewise-constant left-continuous functions of  $k$  which jump no more than by some constant factor  $\nu_0^2 > 1$  (see assumption (a) in Sect. 2.1), when domain  $\Omega_A^B[k]$  (and, simultaneously,  $\Omega_B^A(k^{-1})$ ) changes.

We will fix  $\kappa_0$  at the end of the proof, but in any case it will be less than 1. Since  $Z \leq 1$ , Step 2 ensures that  $Z_A(1), Z_B(1) \leq \zeta_0$  for some absolute constant  $\zeta_0$  (actually,  $Z_A(1)$  and  $Z_B(1)$  are much smaller, being of order  $Z^{1/2}$ ). Now let us start to increase the parameter  $k$ . Since  $\Omega_A^B[k'] \subset \Omega_A^B[k]$  for  $k' > k$ , the partition function  $Z_A(k)$  increases, while  $Z_B(k)$  decreases. Let

$$k_{\max} := \max\{k \geq 1 : Z_A(k) \leq \zeta_0\}.$$

Due to Step 1, there exists a positive constant  $c_0 \leq 1$  such that the following is fulfilled:

$$c_0 k \leq Z_A(k)/Z_B(k) \leq c_0^{-1} k \quad \text{and} \quad c_0 Z \leq Z_A(k)Z_B(k) \leq c_0^{-1} Z$$

for any  $k \in [1, k_{\max}]$ . Moreover, one has  $Z_A(k_{\max}) \geq \nu_0^{-2} \zeta_0$ , since the function  $Z_A(\cdot)$  cannot jump too much at the point  $k_{\max}$ . Therefore, we obtain the estimate

$$k_{\max} \geq c_0 \cdot \frac{Z_A(k_{\max})}{Z_B(k_{\max})} \geq c_0^2 \cdot \frac{(Z_A(k_{\max}))^2}{Z} \geq \nu_0^{-4} \zeta_0^2 c_0^2 \cdot Z^{-1}.$$

Thus, provided that  $\kappa_0 \leq \min\{1, \nu_0^{-4} \zeta_0^2 c_0^2\}$ , (ii) holds true for all  $k \in [1; \kappa_0 Z^{-1}]$  (and similar arguments can be applied for  $k \in [\kappa_0^{-1} Z; 1]$ ).

Finally, for all vertices near  $A$ , one has  $Z_\Omega(\cdot; A) \geq \text{const}$  and  $Z \leq \text{const} \cdot Z_\Omega(\cdot; B)$ . Thus, choosing  $\kappa_0$  small enough (independently of  $\Omega, A, B$ ), one ensures that  $\Omega_A^B[\kappa_0 Z^{-1}]$  is connected (and so  $\Omega_A^B[k]$  is connected for all  $k \geq \kappa_0 Z^{-1}$  as well).  $\square$

Dealing with more involved configurations (e.g. simply connected discrete domains with many marked boundary points, in addition to Theorem 5.1, it is useful to have some information concerning mutual “topological” properties of cross-cuts separating  $A$  and  $B$ , corresponding to *different* pairs  $A, B$ . In order to shorten the notation below, for  $x \in \partial\Omega \setminus (A \cup B)$ , we set

$$\Omega_A^B[x] := \Omega_A^B[R_\Omega(x; A, B)] = \left\{ u \in \Omega : \frac{Z_\Omega(u; A)}{Z_\Omega(u; B)} \geq \frac{Z_\Omega(x; A)}{Z_\Omega(x; B)} \right\}.$$

Roughly speaking,  $\Omega_A^B[x]$  is the set of those  $u \in \Omega$  which are “not further in  $\Omega$ ” from  $A$  comparing to  $B$  than a reference point  $x$ . Note that, since the function  $R_\Omega(\cdot; A, B)$  is monotone on the boundary arcs  $(a_2 b_1)_\Omega$  and  $(b_2 a_1)_\Omega$  (see Lemma 4.1),  $\Omega_A^B[x]$  also behaves in a monotone way when  $x$  runs along  $\partial\Omega \setminus (A \cup B)$ .

**Proposition 5.2.** *Let  $\Omega$  be a simply connected discrete domain, disjoint boundary arcs  $A = [a_1 a_2]_\Omega$ ,  $B = [b_1 b_2]_\Omega$  and  $C = [c_1 c_2]_\Omega$  be listed counterclockwise, and  $B \cup C = [b_1 c_2]_\Omega$  (i.e.,  $b_2$  and  $c_1$  are consecutive points of  $\partial\Omega$ , see Fig. 2B). Then,*

$$\begin{aligned} \Omega_A^C[x] &\subset \Omega_A^{B \cup C}[x] \subset \Omega_A^B[x] \quad \text{for any } x \in (a_2 b_1)_\Omega, \\ \Omega_A^B[y] &\subset \Omega_A^{B \cup C}[y] \subset \Omega_A^C[y] \quad \text{for any } y \in (b_2 a_1)_\Omega. \end{aligned}$$

*Proof.* Let  $x \in (a_2 b_1)_\Omega$  (the case  $y \in (b_2 a_1)_\Omega$  is similar) and  $u \in \text{Int } \Omega_A^C[x]$ . We need to check that  $u \in \text{Int } \Omega_A^{B \cup C}[x]$ . By definition,

$$\begin{aligned} u \in \text{Int } \Omega_A^C[x] &\Leftrightarrow Z_\Omega(u; A) \cdot Z_\Omega(x; C) \geq Z_\Omega(x; A) \cdot Z_\Omega(u; C), \\ u \in \text{Int } \Omega_A^{B \cup C}[x] &\Leftrightarrow Z_\Omega(u; A) \cdot Z_\Omega(x; B \cup C) \geq Z_\Omega(x; A) \cdot Z_\Omega(u; B \cup C). \end{aligned} \tag{5.10}$$

Since  $Z_\Omega(\cdot; B \cup C) = Z_\Omega(\cdot; B) + Z_\Omega(\cdot; C)$ , it is sufficient to prove that, for any  $b \in B$ ,

$$\frac{Z_\Omega(x; b)}{Z_\Omega(u; b)} = \frac{Z_\Omega(x; b_{\text{int}})}{Z_\Omega(u; b_{\text{int}})} \geq \frac{Z_\Omega(x; A)}{Z_\Omega(u; A)}.$$

For  $v \in \Omega$ , denote

$$H(v) := \begin{cases} Z_\Omega(u; A) \cdot Z_\Omega(x; v) - Z_\Omega(x; A) \cdot Z_\Omega(u; v), & v \in \text{Int } \Omega, \\ \mu_x^{-1} \mathbb{1}[v=x], & v \in \partial\Omega, \end{cases}$$

Suppose that, on the contrary,  $H(b_{\text{int}}) < 0$  for some  $b \in B$ . Since the function  $H$  is harmonic everywhere in  $\Omega$  except  $u$  (where it is subharmonic), and vanishes on  $\partial\Omega$  everywhere except  $x$  (where it is strictly positive), there exists a nearest-neighbor path  $\gamma_{bu}$  running from  $b_{\text{int}}$  to  $u$  such that  $H < 0$  along  $\gamma_{bu}$ . On the other hand,  $H(c_{\text{int}}) \geq 0$  for at least one  $c \in C$  (otherwise, summation along the arc  $C$  gives a contradiction with the first part of (5.10)). Hence, there exists a nearest-neighbor path  $\gamma_{cx}$  running from  $c_{\text{int}}$  to  $x$  such that  $H \geq 0$  along  $\gamma_{cx}$ . Since these two paths cannot cross each other and  $\Omega$  is simply connected,  $\gamma_{cx}$  should separate  $u$  and  $A$ . Then, the maximum principle implies  $H(a_{\text{int}}) > 0$  for any  $a \in A$ . Summing along the arc  $A$ , one arrives at the inequality

$$Z_\Omega(u; A) \cdot Z_\Omega(x; A) > Z_\Omega(x; A) \cdot Z_\Omega(u; A), \quad (5.11)$$

which is a contradiction.

Now let  $u \in \Omega_A^{B \cup C}[x]$ . Arguing as above, in order to deduce  $u \in \Omega_A^B[x]$  from the second part of (5.10), it is sufficient to prove that, for all  $c \in C$ ,

$$\frac{Z_\Omega(x; c)}{Z_\Omega(u; c)} = \frac{Z_\Omega(x; c_{\text{int}})}{Z_\Omega(u; c_{\text{int}})} \leq \frac{Z_\Omega(x; A)}{Z_\Omega(u; A)}.$$

Suppose, on the contrary, that  $H(c_{\text{int}}) > 0$  for some  $c \in C$ . Then there exists a path  $\gamma_{cx}$  running from  $c_{\text{int}}$  to  $x$  such that  $H > 0$  along  $\gamma_{cx}$ . Now there are two cases. If  $\gamma_{cx}$  separates  $u$  and  $A$ , then the maximum principle implies  $H(a_{\text{int}}) > 0$  for all  $a \in A$ , which leads to the same contradiction (5.11). But if  $\gamma_{cx}$  does not separate  $u$  and  $A$ , then it separates  $u$  and  $B$ . Therefore,  $H(b_{\text{int}}) > 0$  for all  $b \in B$ , which directly gives  $u \in \Omega_A^B[x]$  by summation along  $B$ .  $\square$

## 6. EXTREMAL LENGTHS

In this Section we recall the notion of a discrete extremal length  $L_\Omega([ab]_\Omega; [cd]_\Omega)$  between two opposite boundary arcs of a discrete simply connected domain  $\Omega$  (which is nothing but the resistance of the corresponding electrical network), firstly discussed by Duffin in [Duf62]. Note that  $L_\Omega$  can be defined in two *equivalent* ways: (a) via some extremal problem (see Definition 6.1), and (b) via solution to a Dirichlet-Neumann boundary value problem (see Proposition 6.4 and Remark 6.5). We also refer the reader to [BV12], where this equivalence and a connection to the Uniform Spanning Tree model are discussed in more details. The most important feature of (a) is that it allows one to estimate  $L_\Omega$  “in geometric terms”. In particular, we show that  $L_\Omega$  is uniformly comparable to its continuous counterpart – extremal length of the corresponding polygonal quadrilateral (see Proposition 6.2 and Corollary 6.3 for details). At the same time, approach (b) allows us to relate  $L_\Omega$  to the random walk partition function  $Z_\Omega$  discussed above (see Proposition 6.6). Note that this connection is of



crucial importance for the next Section, which starts with the complete set of uniform double-sided estimates relating  $Y_\Omega$ ,  $Z_\Omega$  and  $L_\Omega$  (see Theorem 7.1).

Let  $\Omega$  be a discrete domain and  $E^\Omega = E_{\text{int}}^\Omega \cup E_{\text{bd}}^\Omega$  be the set of edges of  $\Omega$ . For a given function (“discrete metric”)  $g : E^\Omega \rightarrow [0; +\infty)$  we define the “ $g$ -area” of  $\Omega$  by

$$A_g(\Omega) := \sum_{e \in E^\Omega} w_e g_e^2,$$

where  $w_e$  denote weights of edges of  $\Gamma$  (see Section 2.1). Further, for a given subset  $\gamma \subset E^\Omega$  (e.g., a nearest-neighbor path running in  $\Omega$ ), we define its “ $g$ -length” by

$$L_g(\gamma) := \sum_{e \in \gamma} g_e.$$

Finally, for a family  $\mathcal{E}$  of lattice paths in  $\Omega$ , we set  $L_g(\mathcal{E}) := \inf_{\gamma \in \mathcal{E}} L_g(\gamma)$ .

**Definition 6.1.** *The discrete extremal length of the family  $\mathcal{E}$  is given by*

$$L[\mathcal{E}] := \sup_{g: E^\Omega \rightarrow [0; +\infty)} \frac{[L_g(\mathcal{E})]^2}{A_g(\Omega)}, \quad (6.1)$$

where the supremum is taken over all  $g$ 's such that  $0 < A_g(\Omega) < +\infty$ . In particular, if  $\Omega$  is simply connected,  $a, b, c, d \in \partial\Omega$  are listed counterclockwise, and  $b \neq c$ ,  $d \neq a$ , then we define  $L_\Omega([ab]_\Omega; [cd]_\Omega)$  as the extremal length of the family  $(\Omega; [ab]_\Omega \leftrightarrow [cd]_\Omega)$  of all lattice paths connecting the boundary arcs  $[ab]_\Omega$  and  $[cd]_\Omega$  inside  $\Omega$ .

Definition 6.1 easily allows one to estimate the extremal length  $L_\Omega([ab]_\Omega; [cd]_\Omega)$  from below, since for this purpose it is sufficient to take any “discrete metric”  $g$  in  $\Omega$  and estimate  $A_g(\Omega)$  and  $L_g(\Omega; [ab]_\Omega \leftrightarrow [cd]_\Omega)$  for this particular  $g$ . Note that the most natural way to give an upper bound is to use (some form of) duality (6.6).

For a (simply connected) discrete domain  $\Omega \subset \Gamma$  we denote its **polygonal representation** as the open (simply connected) set  $\Omega^C \subset \mathbb{C}$  bounded by the polyline  $x_{\text{mid}}^0 x_{\text{mid}}^1 \dots x_{\text{mid}}^n x_{\text{mid}}^0$  passing through all *middle points*  $x_{\text{mid}}^k := \frac{1}{2}(x^k + x_{\text{int}}^k)$  of boundary edges  $(x_{\text{int}}^k, x^k) \in \partial\Omega$  in their natural order (counterclockwise with respect to  $\Omega$ ), see Fig. 1. For  $a, b \in \partial\Omega$ ,  $a \neq b$ , we denote by  $[ab]_\Omega^C \subset \partial\Omega^C$  the part of this polyline from  $a_{\text{mid}}$  to  $b_{\text{mid}}$ , viewed as a boundary arc of  $\Omega^C$ . In case  $a = b$ , we slightly modify this definition, setting, say,  $[aa]_\Omega^C := [\frac{1}{2}(a_{\text{mid}}^- + a_{\text{mid}}); a_{\text{mid}}] \cup [a_{\text{mid}}; \frac{1}{2}(a_{\text{mid}}^+ + a_{\text{mid}})]$ , where  $a^\pm$  denote the boundary points of  $\Omega$  just before and next to  $a$ .

Let  $L_\Omega^C := L_\Omega^C([ab]_\Omega^C; [cd]_\Omega^C)$  denote the classical extremal distance between the opposite arcs of a topological quadrilateral  $(\Omega^C; a_{\text{mid}}, b_{\text{mid}}, c_{\text{mid}}, d_{\text{mid}})$  in the complex plane (e.g., see [Ahl73, Chapter 4] or [GM05, Chapter IV]). Note that our Definition 6.1 replicates the classical one, which says

$$L_\Omega^C([ab]_\Omega^C; [cd]_\Omega^C) = \sup_{g: \Omega^C \rightarrow [0; +\infty)} \frac{\left[ \inf_{\gamma: [ab]_\Omega^C \leftrightarrow [cd]_\Omega^C} \int_\gamma g ds \right]^2}{\iint_\Omega g^2 dx dy}, \quad (6.2)$$

where the supremum is taken over all  $g$  such that  $0 < \iint_\Omega g^2 dx dy < +\infty$  and the infimum is over all curves connecting  $[ab]_\Omega^C$  and  $[cd]_\Omega^C$  inside  $\Omega^C$  (see [Ahl73, GM05]).

**Proposition 6.2.** *Let  $\Omega$  be a simply connected discrete domain and  $a, b, c, d \in \partial\Omega$ ,  $b \neq c$ ,  $d \neq a$ , be listed in the counterclockwise order. Then,*

$$L_\Omega([ab]_\Omega; [cd]_\Omega) \asymp L_\Omega^C([ab]_\Omega^C; [cd]_\Omega^C) \quad (6.3)$$

with some absolute (i.e., independent of  $\Omega, a, b, c, d$ ) constants.

*Proof.* Let  $L_{\text{disc}} := L_{\Omega}([ab]_{\Omega}; [cd]_{\Omega})$  and  $L_{\text{cont}} := L_{\Omega}^{\mathbb{C}}([ab]_{\Omega}^{\mathbb{C}}; [cd]_{\Omega}^{\mathbb{C}})$ . We prove two one-sided estimates separately, taking a solution to either discrete (6.1) or continuous (6.2) extremal problem, and constructing some related metric for the other one, thus obtaining a lower bound for the other (continuous or discrete) extremal length.

(i)  $L_{\text{cont}} \geq \text{const} \cdot L_{\text{disc}}$ . Let  $g_e^{\max}$ ,  $e \in E^{\Omega}$ , be the extremal metric in (6.1). For a face  $f$  of  $\Gamma$  (considered as a convex polygon in  $\mathbb{C}$ ), let  $\Lambda_f \subset \Gamma$  be defined by saying that  $\text{Int } \Lambda_f$  consists of all vertices incident to  $f$ , and  $\Lambda_f^{\mathbb{C}}$  be the polygonal representation of  $\Lambda_f$ . Further, for an edge  $e \in E^{\Gamma}$  separating two faces  $f$  and  $f'$ , let  $\text{Int } \Lambda_e := \text{Int } \Lambda_f \cup \text{Int } \Lambda_{f'}$  and  $\Lambda_e^{\mathbb{C}}$  be the polygonal representation of  $\Lambda_e$  (see Fig. 1B). We set

$$g(z) := \sum_{e \in E^{\Omega}} g_e^{\max} r_e^{-1} \mathbb{1}_{\Lambda_e^{\mathbb{C}}}(z), \quad z \in \Omega^{\mathbb{C}},$$

where  $r_e$  denotes the length of  $e$ . Since each point in  $\Omega^{\mathbb{C}}$  belongs to a uniformly bounded number of edge neighborhoods  $\Lambda_e^{\mathbb{C}}$  (recall that degrees of faces and vertices of  $\Gamma$  are uniformly bounded), one has

$$\iint_{\Omega} g^2 dx dy \asymp \sum_{e \in E^{\Omega}} (g_e^{\max})^2 r_e^{-2} \text{Area}(\Lambda_e^{\mathbb{C}} \cap \Omega^{\mathbb{C}}) \asymp \sum_{e \in E^{\Omega}} w_e (g_e^{\max})^2, \quad (6.4)$$

as  $r_e^{-2} \text{Area}(\Lambda_e^{\mathbb{C}} \cap \Omega^{\mathbb{C}}) \asymp 1 \asymp w_e$  (see our assumptions on  $\Gamma$  listed in Sect. 2.1).

Now let  $\gamma$  be any continuous curve crossing  $\Omega^{\mathbb{C}}$  from  $[ab]_{\Omega}^{\mathbb{C}}$  to  $[cd]_{\Omega}^{\mathbb{C}}$ ,  $F^{\gamma}$  be the set of all (closed) faces touched by  $\gamma$ , and  $E^{\gamma} \subset E^{\Omega}$  be the set of all edges of  $\Omega$  incident to those faces. It is clear that  $E^{\gamma}$  contains a discrete nearest-neighbor path running from  $[ab]_{\Omega}$  to  $[cd]_{\Omega}$ . Thus, it is sufficient to estimate  $\int_{\gamma} g ds$  (from below) via  $\sum_{e \in E^{\gamma}} g_e^{\max}$ . Note that, for any  $f \in F^{\gamma}$ ,

*$\gamma$  should cross the annulus type polygon  $\Lambda_f^{\mathbb{C}} \setminus f$  at least once.*

Let  $\gamma_f$  denote this crossing (there is one exceptional situation: if, say,  $b$  and  $c$  are two consecutive boundary points and  $f$  is a boundary face between them, then  $\gamma$  may not cross the annulus  $\Lambda_f^{\mathbb{C}} \setminus f$ , so we denote by  $\gamma_f$  be the corresponding crossing of  $\Lambda_f^{\mathbb{C}}$  itself). As degrees of vertices and faces of  $\Gamma$  are uniformly bounded, each piece of  $\gamma$  belongs to a bounded number of  $\gamma_f$ . Since  $\text{Length}(\gamma_f) \geq \text{const} \cdot r_e$  for any  $e \sim f$  (all those  $r_e$  are comparable to each other due to our assumptions), we arrive at

$$\int_{\gamma} g ds \geq \text{const} \cdot \sum_{f \in F^{\gamma}} \int_{\gamma_f} g ds \geq \text{const} \cdot \sum_{e \sim f \in F^{\gamma}} \text{Length}(\gamma_f) g_e^{\max} r_e^{-1} \geq \text{const} \cdot \sum_{e \in E^{\gamma}} g_e^{\max}.$$

Together with (6.4), this allows us to conclude that

$$L_{\text{cont}} \geq \frac{[\inf_{\gamma} \int_{\gamma} g ds]^2}{\iint_{\Omega} g^2 dx dy} \geq \text{const} \cdot \frac{[\inf_{\gamma} L_{g^{\max}}(E^{\gamma})]^2}{A_{g^{\max}}(\Omega)} \geq \text{const} \cdot L_{\text{disc}}.$$

(ii)  $L_{\text{disc}} \geq \text{const} \cdot L_{\text{cont}}$ . Let  $g^{\max} : \Omega^{\mathbb{C}} \rightarrow \mathbb{R}^+$  be the extremal metric in (6.2). It is well known that  $g^{\max}(z) \equiv |\phi'(z)|$  where  $\phi$  conformally maps  $\Omega^{\mathbb{C}}$  onto the rectangle:

$$\begin{aligned} \phi : \Omega^{\mathbb{C}} &\rightarrow \{z : 0 < \text{Re } z < 1, 0 < \text{Im } z < L_{\text{cont}}^{-1}\}, \\ a &\mapsto iL_{\text{cont}}^{-1}, \quad b \mapsto 0, \quad c \mapsto 1, \quad d \mapsto 1 + iL_{\text{cont}}^{-1}. \end{aligned} \quad (6.5)$$

We set

$$g_e := \int_{\Omega^c \cap e} g^{\max} ds, \quad e \in E^\Omega.$$

Note that, for each nearest-neighbor discrete path  $\gamma$  in  $\Omega$ , we have  $\sum_{e \in \gamma} g_e = \int_\gamma g^{\max} ds$ , thus it is sufficient to estimate  $\sum_{e \in \Omega_e} w_e g_e^2$  (from above) via  $\iint (g^{\max})^2 dx dy$ .

Let  $z_e$  denote the mid-point of an *inner* edge  $e$ . As  $\phi$  is a univalent holomorphic function (in  $\Lambda_e^c \cap \Omega^c$ ), all values  $|\phi'(z)|$  for  $z \in e$  are uniformly comparable to each other (and comparable to all other values  $|\phi'(z)|$  for  $z$  near  $z_e$ ), e.g., see [Ahl73, Chapter 5, Theorem 5-3] or [GM05, Chapter 1, Theorem 4.5]. In particular, this implies

$$g_e^2 \asymp r_e^2 |\phi'(z_e)|^2 \leq \text{const} \cdot \iint_{\Lambda_e^c \cap \Omega^c} |\phi'|^2 dx dy.$$

The same holds true for *boundary* edges: if  $\Omega^c$  has an inner angle  $\theta_x \in (\eta_0; 2\pi]$  at the boundary point  $x_{\text{mid}} \in \partial\Omega$ , then  $\phi$  behaves like  $(z - x_{\text{mid}})^{\pi/\theta_x}$  near  $x$  (or  $(z - x_{\text{mid}})^{\pi/2\theta_x}$ , if  $x$  is one of the corners  $a, b, c, d$ ), hence  $|\phi'|$  blows up not faster than  $|z - x_{\text{mid}}|^{-1/2}$  (or  $|z - x_{\text{mid}}|^{-3/4}$ , respectively) when  $z$  approaches  $x_{\text{mid}}$ , which means  $g_e \asymp r_e |\phi'(x_{\text{int}})|$ .

As each point in  $\Omega^c$  belongs to a uniformly bounded number of  $\Lambda_e^c$ , we obtain

$$\sum_{e \in E^\Omega} w_e g_e^2 \leq \text{const} \cdot \iint_{\Omega^c} |\phi'|^2 dx dy.$$

Therefore,

$$L_{\text{disc}} \geq \frac{[\inf_\gamma \sum_{e \in \gamma} g_e]^2}{\sum_{e \in E^\Omega} w_e g_e^2} \geq \text{const} \cdot \frac{[\inf_\gamma \int_\gamma g^{\max} ds]^2}{\iint_{\Omega^c} (g^{\max})^2 dx dy} \geq \text{const} \cdot L_{\text{cont}}. \quad \square$$

**Corollary 6.3.** *Let  $\Omega$  be a simply connected discrete domain and  $a, b, c, d \in \partial\Omega$  be four distinct boundary points listed counterclockwise. Then,*

$$L_\Omega([ab]_\Omega; [cd]_\Omega) \cdot L_\Omega([bc]_\Omega; [da]_\Omega) \asymp 1 \quad (6.6)$$

*with some absolute (i.e., independent of  $\Omega, a, b, c, d$ ) constants.*

*Proof.* Directly follows from (6.3) applied to both factors and the *exact* duality of continuous extremal lengths:  $L_\Omega^c([ab]_\Omega^c; [cd]_\Omega^c) \cdot L_\Omega^c([bc]_\Omega^c; [da]_\Omega^c) = 1$ .  $\square$

We now pass to the second approach to the notion of extremal length via solution to the following Dirichlet-Neumann boundary value problem (which corresponds to the real part  $\text{Re } \phi$  of the uniformization map (6.5)):

Let  $\Omega$  be simply connected and  $a, b, c, d \in \partial\Omega$ ,  $b \neq c$ ,  $d \neq a$ , be listed counterclockwise. Denote by  $V = V_{\Omega; [ab]_\Omega, [cd]_\Omega} : \Omega \rightarrow [0; 1]$  the unique discrete harmonic in  $\Omega$  function (electric potential) such that  $V \equiv 0$  on  $[ab]_\Omega$ ,  $V \equiv 1$  on  $[cd]_\Omega$ , and  $V$  satisfies Neumann boundary conditions (i.e.,  $V(x_{\text{int}}) = V(x)$ ) for  $x \in \partial\Omega \setminus ([ab]_\Omega \cup [cd]_\Omega)$ . We also set

$$I(V) := \sum_{x \in [ab]_\Omega} w_{xx_{\text{int}}} V(x_{\text{int}}) = \sum_{x \in [cd]_\Omega} w_{xx_{\text{int}}} (1 - V(x_{\text{int}}))$$

(note that  $\sum_{x \in \partial\Omega} w_{xx_{\text{int}}} (V(x) - V(x_{\text{int}})) = \sum_{u \in \text{Int } \Omega} \mu_u [\Delta V](u) = 0$ ).

The next Proposition rephrase  $L_\Omega([ab]_\Omega; [cd]_\Omega)$  via  $I(V)$  (which is nothing but the electric current in the corresponding network). Note that, on the contrary to the classical setup, (6.7) does *not* allow to replace double-sided estimate (6.6) by the identity. Indeed, mimicking the continuous case, one can pass from  $V$  to its harmonic conjugate function  $V^*$  that solves the similar boundary value problem for dual arcs, but this  $V^*$  is defined on a dual graph  $\Gamma^*$ , leading to the extremal length of some *other* discrete quadrilateral (drawn on  $\Gamma^*$ ) rather than  $\Omega \subset \Gamma$  itself (see Remark 6.5).

**Proposition 6.4.** *For any simply connected discrete domain  $\Omega$  and any  $a, b, c, d \in \partial\Omega$ ,  $b \neq c$ ,  $d \neq a$ , listed counterclockwise, the following is fulfilled:*

$$L_\Omega([ab]_\Omega; [cd]_\Omega) = [I(V_{(\Omega; [ab]_\Omega, [cd]_\Omega)})]^{-1}. \quad (6.7)$$

*Proof.* See [Duf62] and [Vor10, BV12]. The core idea is to construct the function  $V$  using the extremal “discrete metric”  $g^{\max}$  for the family  $\mathcal{E} := (\Omega; [ab]_\Omega \leftrightarrow [cd]_\Omega)$  (which is unique up to a multiplicative constant due to simple convexity reasons). Namely, let  $(\Omega; u \leftrightarrow [ab]_\Omega)$  denote the family of all discrete paths running from  $u \in \Omega$  to the boundary arc  $[ab]_\Omega$  inside  $\Omega$ , and

$$V(u) := L_{g^{\max}}(\Omega; u \leftrightarrow [ab]_\Omega).$$

Then,  $V$  is constant on  $[cd]_\Omega$  and satisfies Neumann boundary conditions on  $(bc)_\Omega$ ,  $(da)_\Omega$  (if one of these properties fails, then one can improve  $g^{\max}$  on the corresponding boundary edge so that  $L_g(\mathcal{E})$  does not change while  $A_g(\Omega)$  decreases). In particular, one can normalize  $g^{\max}$  so that  $V \equiv 1$  on  $[cd]_\Omega$ .

Moreover,  $V$  is discrete harmonic in  $\Omega$ . Indeed, note that  $V(u') - V(u) = \pm g_{uu'}^{\max}$  for any  $(uu') \in E^\Omega$  (otherwise, one can improve  $g_{uu'}^{\max}$ ). Then, for a given  $u \in \text{Int } \Omega$ , replacing  $g_{uu'}^{\max}$  by  $g_{uu'}^{\max} + \varepsilon$  on all edges  $(uu') \in E^\Omega$  such that  $V(u') > V(u)$  and, simultaneously, replacing  $g_{uu'}^{\max}$  by  $g_{uu'}^{\max} - \varepsilon$  on all  $(uu') \in E^\Omega$  such that  $V(u') < V(u)$ , one does not change any global distances (and, in particular, does not change  $L_g(\mathcal{E})$ ), while the area  $A_g(\Omega)$  changes by  $\varepsilon \mu_u[\Delta V](u) + O(\varepsilon^2)$ .

Finally, using discrete integration by parts and  $[\Delta V](u) \equiv 0$ , one concludes that

$$\begin{aligned} L_\Omega^{-1} &= A_{g^{\max}}(\Omega) = \sum_{e=(uu') \in E^\Omega} w_e (V(u') - V(u))^2 \\ &= - \sum_{u \in \text{Int } \Omega} \mu_u [\Delta V](u) V(u) - \sum_{x \in \partial\Omega} w_{xx_{\text{int}}} (V(x_{\text{int}}) - V(x)) V(x) \\ &= \sum_{x \in [cd]_\Omega} w_{xx_{\text{int}}} (1 - V(x_{\text{int}})) = I(V). \end{aligned} \quad \square$$

Note that, for any discrete harmonic in  $\Omega$  function  $V$ , one can construct a discrete *harmonic conjugate* function  $V^*$  which is uniquely defined (up to an additive constant) on faces of  $\Omega$  (including boundary ones) by saying

$$H(f_{vv'}^{\text{left}}) - H(f_{vv'}^{\text{right}}) := w_{vv'} \cdot (H(v') - H(v)) \quad (6.8)$$

for any oriented edge  $(vv') \in E^\Omega$ , where  $f_{vv'}^{\text{left}}$  and  $f_{vv'}^{\text{right}}$  denote faces to the left and to the right of  $(vv')$ , respectively. The function  $V^*$  is well defined locally (iff  $\Delta V = 0$ ), and hence well defined globally, as  $\Omega$  is simply connected. Moreover, for any inner face  $f$  in  $\Omega$ , it satisfies a discrete harmonicity condition

$$\sum_{f' \sim f} w_{ff'} (V^*(f') - V^*(f)) = 0, \quad (6.9)$$

where  $w_{ff'} := w_{vv'}^{-1}$  for any couple of dual edges  $(ff') = (vv')^*$ .

**Remark 6.5.** If one takes  $V = V_{(\Omega; [ab]_\Omega, [cd]_\Omega)}$ , then the harmonic conjugate function  $V^*$  is constant along boundary arcs  $(bc)_\Omega$  and  $(da)_\Omega$  (since  $V$  satisfies Neumann boundary conditions on these arcs). Fixing an additive constant so that  $V^* \equiv 0$  on  $(bc)_\Omega$  and tracking the increment of  $V^*$  along  $[ab]_\Omega$ , one obtains  $V^* \equiv I(V)$  on  $(da)_\Omega$ . Further, Dirichlet boundary conditions for  $V$  on  $[ab]_\Omega$  and  $[cd]_\Omega$  can be directly translated into Neumann ones for  $V^*$  (one can easily see that  $V^*$  satisfies (6.9) with smaller number of terms at all faces touching  $[ab]_\Omega$  or  $[cd]_\Omega$ ). Thus,  $[I(V)]^{-1} \cdot V^*$  solves the same Dirichlet-Neumann boundary value problem for the *dual* quadrilateral drawn on  $\Gamma^*$ . Moreover,

$$\sum_{(ff')^* \in E^\Omega} w_{ff'} (V^*(f') - V^*(f))^2 = \sum_{(vv') \in E^\Omega} w_{vv'} (V(v') - V(v))^2 = L_\Omega^{-1},$$

and hence the corresponding dual extremal length  $L_\Omega^*$  is equal to  $[I(V)^{-2} L_\Omega^{-1}]^{-1} = L_\Omega^{-1}$ .

The last Proposition in this Section gives an estimate for the partition function  $Z_\Omega([ab]_\Omega; [cd]_\Omega)$  of random walks joining  $[ab]_\Omega$  and  $[cd]_\Omega$  in  $\Omega$  via the extremal length  $L_\Omega([ab]_\Omega; [cd]_\Omega)$  (note that the latter can be thought about as the (reciprocal of) similar partition function for random walks *reflecting* from the dual boundary arcs).

**Proposition 6.6.** *Let  $\Omega$  be a simply connected discrete domain, and  $a, b, c, d \in \partial\Omega$ ,  $b \neq c$ ,  $d \neq a$ , be listed counterclockwise. Then, the following is fulfilled:*

$$Z_\Omega([ab]_\Omega; [cd]_\Omega) \leq \text{const} \cdot (L_\Omega([ab]_\Omega; [cd]_\Omega))^{-1},$$

where *const* does not depend on  $\Omega, a, b, c, d$ . Moreover, if  $L_\Omega([ab]_\Omega; [cd]_\Omega) \leq \text{const}$ , then

$$Z_\Omega([ab]_\Omega; [cd]_\Omega) \asymp (L_\Omega([ab]_\Omega; [cd]_\Omega))^{-1}$$

(with constants depending on the upper bound for  $L_\Omega$  but independent of  $\Omega, a, b, c, d$ ).

*Proof.* It is easy to see that, for any  $u \in \text{Int } \Omega$ ,  $V(u)$  is equal to the probability of the event that the random walk started at  $u$  and reflecting from complementary arcs  $(bc)_\Omega, (da)_\Omega$  exists  $\Omega$  through  $[cd]_\Omega$  (indeed, this probability is a discrete harmonic function which satisfies the same boundary conditions as  $V$ ). Hence, for any  $x \in [ab]_\Omega$ ,

$$V(x_{\text{int}}) \geq \text{const} \cdot Z_\Omega(x_{\text{int}}; [cd]_\Omega) \asymp Z_\Omega(x; [cd]_\Omega),$$

since the right-hand side is (up to constant) the same probability for the random walk with absorbing boundary conditions on  $(bc)_\Omega$  and  $(da)_\Omega$ . Thus, (6.7) gives

$$(L_\Omega([ab]_\Omega; [cd]_\Omega))^{-1} = \sum_{x \in [ab]_\Omega} w_{xx_{\text{int}}} V(x_{\text{int}}) \geq \text{const} \cdot Z_\Omega([ab]_\Omega; [cd]_\Omega).$$

Further, let  $L_\Omega := L_\Omega([ab]_\Omega; [cd]_\Omega) \leq \text{const}$ . Due to duality given by Corollary 6.3, it is equivalent to  $L_\Omega([bc]_\Omega; [da]_\Omega) \geq \text{const}$ . We have seen above that this implies

$$Z_\Omega([bc]_\Omega; [da]_\Omega) \leq \text{const}$$

which is equivalent to  $Y_\Omega(b, c; d, a) \leq \text{const}$  due to Theorem 4.8. Therefore,

$$Z_\Omega := Z_\Omega([ab]_\Omega; [cd]_\Omega) \asymp \log(1 + Y_\Omega(a, b; c, d)) = \log(1 + (Y_\Omega(b, c; d, a))^{-1}) \geq \text{const}.$$

Since  $Z_\Omega \leq \text{const} \cdot L_\Omega^{-1}$  in any case, this implies  $Z_\Omega \asymp 1$ , if  $L_\Omega \asymp 1$ .

Thus, we are mostly interested in the situation when  $L_\Omega$  is very small (i.e., boundary arcs  $[ab]_\Omega$  and  $[cd]_\Omega$  are “very close” to each other in  $\Omega$ ). Our strategy in this case is similar to the proof of Theorem 4.8: we split  $[ab]_\Omega$  into several smaller pieces  $[a_k a_{k+1}]_\Omega$

such that  $L_\Omega([a_k a_{k+1}]_\Omega; [cd]_\Omega) \asymp 1$  and apply the result obtained above to each of these smaller arcs. Recall that

$$L_\Omega^{-1} = \sum_{x \in [ab]_\Omega} w_{xx_{\text{int}}} V(x_{\text{int}}).$$

We construct the boundary points  $a = a_0, a_1, \dots, a_{n+1} = b \in \partial\Omega$  inductively by the following procedure: if  $a_k$  is already chosen, we move  $a_{k+1}$  further along the boundary arc  $[ab]_\Omega$  step by step until the first vertex  $a_{k+1}$  such that

$$(L_\Omega([a_k a_{k+1}]_\Omega; [cd]_\Omega))^{-1} = \sum_{x \in [a_k a_{k+1}]_\Omega} w_{xx_{\text{int}}} V_{(\Omega; [a_k a_{k+1}]_\Omega, [cd]_\Omega)}(x_{\text{int}}) \geq 1$$

(or  $a_{k+1} = b$ ). Note that this sum cannot increase by more than some absolute constant  $\nu_0$  on each step (as we increase the absorbing boundary  $[a_k a_{k+1}]_\Omega$ , all terms decreases, while the new (last) term is no greater than  $w_{xx_{\text{int}}} \leq \nu_0$ ). Therefore,  $L_\Omega([a_k a_{k+1}]_\Omega; [cd]_\Omega) \asymp 1$  for all  $k$ , possibly except the last one (when we are forced to choose  $a_{n+1} = b$  before the sum becomes large). As we have seen above, this implies

$$Z_\Omega([a_k a_{k+1}]_\Omega; [cd]_\Omega) \asymp 1 \quad \text{for all } k = 0, 1, \dots, n-1.$$

Note that  $V = V_{(\Omega; [ab]_\Omega, [cd]_\Omega)} \leq V_{(\Omega; [a_k a_{k+1}]_\Omega, [cd]_\Omega)}$  due to monotonicity of boundary conditions (the absorbing boundary is larger in the first case), which gives

$$\sum_{x \in [a_k a_{k+1}]_\Omega} w_{xx_{\text{int}}} V(x_{\text{int}}) \leq \sum_{x \in [a_k a_{k+1}]_\Omega} w_{xx_{\text{int}}} V_{(\Omega; [a_k a_{k+1}]_\Omega, [cd]_\Omega)}(x_{\text{int}}) \leq \text{const}$$

for all  $k = 0, 1, \dots, n$ , thus  $L^{-1} \leq \text{const} \cdot (n+1) \asymp n$ , which implies the inverse estimate

$$Z_\Omega \asymp \sum_{k=0}^n Z_\Omega([a_k a_{k+1}]_\Omega; [cd]_\Omega) \geq \text{const} \cdot n \asymp L_\Omega^{-1}. \quad \square$$

## 7. DOUBLE-SIDED ESTIMATES OF HARMONIC MEASURE.

We start this Section by Theorem 7.1 which combines uniform estimates obtained above for cross-ratios  $Y_\Omega$ , partition functions  $Z_\Omega$  and extremal lengths  $L_\Omega$  of discrete quadrilaterals  $(\Omega; a, b, c, d)$ . Then, we show how tools developed in our paper can be used to obtain exponential double-sided estimates in terms of appropriate extremal lengths for the discrete harmonic measure  $\omega_\Omega(u; [ab]_\Omega)$  of a “far” boundary arc (similar to the classical ones due to Ahlfors, Beurling and going back to Carleman, see [Ahl73, § 4-5, 4-14] and [GM05, § IV.5, IV.6]). The main result is given by Theorem 7.8. In particular, it allows us to obtain a *uniform* double-sided estimate of  $\log \omega_\Omega(u; [ab]_\Omega)$  via  $\log \omega_{\Omega^c}(u; [ab]_\Omega^c)$ , where  $\omega_{\Omega^c}$  denotes the *continuous* harmonic measure in a polygonal representation of  $\Omega$  (see Corollary 7.9). Note that one cannot hope to prove the similar estimate for  $\omega_\Omega(u; [ab]_\Omega)$  itself: dealing with thin fiords, one faces with exponentially small harmonic measures which are highly sensitive to widths of those fiords.

**Theorem 7.1.** *Let  $\Omega$  be a simply connected discrete domain and distinct boundary points  $a, b, c, d \in \partial\Omega$  be listed counterclockwise. Denote*

$$\begin{aligned} Y &:= Y_\Omega(a, b; c, d), & Z &:= Z_\Omega([ab]_\Omega; [cd]_\Omega), & L &:= L_\Omega([ab]_\Omega; [cd]_\Omega), \\ Y' &:= Y_\Omega(b, c; d, a), & Z' &:= Z_\Omega([bc]_\Omega; [da]_\Omega), & L' &:= L_\Omega([bc]_\Omega; [da]_\Omega). \end{aligned}$$

(i) If at least one of the estimates

$$\begin{aligned} Y &\leq \text{const}, & Z &\leq \text{const}, & L &\geq \text{const}, \\ Y' &\geq \text{const}, & Z' &\geq \text{const}, & L' &\leq \text{const} \end{aligned} \quad (7.1)$$

holds true, then all these estimates hold true (with constants depending on the initial bound but independent of  $\Omega, a, b, c, d$ ). Moreover, if at least one of  $Y, Y', Z, Z', L, L'$  is of order 1 (i.e., admits the double-sided estimate  $\asymp 1$ ), then all of them are of order 1.

(ii) If (7.1) holds true, then the following double-sided estimates are fulfilled:

$$Z \asymp Y \quad \text{and} \quad \log(1+Y^{-1}) \asymp L.$$

In particular, there exist some constants  $\beta_{1,2}, C_{1,2} > 0$  such that the uniform estimate

$$C_1 \cdot \exp[-\beta_1 L] \leq Z \leq C_2 \cdot \exp[-\beta_2 L] \quad (7.2)$$

holds true for any discrete quadrilateral  $(\Omega; a, b, c, d)$  satisfying (7.1).

*Proof.* (i) It follows from Theorem 4.8 and Proposition 6.6 that

$$\log(1+Y) \asymp Z \leq \text{const} \cdot L^{-1} \quad \text{and} \quad \log(1+Y') \asymp Z' \leq \text{const} \cdot (L')^{-1}.$$

Moreover,  $YY' = 1$  by definition, and  $LL' \asymp 1$  due to Corollary 6.3. Therefore, one has

$$\begin{aligned} Y &\leq \text{const} &\Leftrightarrow & Z \leq \text{const} &\Leftarrow & L \geq \text{const} \\ &\Updownarrow & & & & \Updownarrow \\ Y' &\geq \text{const} &\Leftrightarrow & Z' \geq \text{const} &\Leftarrow & L' \leq \text{const}, \end{aligned}$$

which gives the equivalence of all six bounds. Interchanging  $Y, Z, L$  and  $Y', Z', L'$ , one obtains the same equivalence of inverse estimates. Thus, if at least one of these quantities is  $\asymp 1$ , then all others are  $\asymp 1$  as well.

(ii) Since  $Y \leq \text{const}$ , Remark 4.6 guarantees that  $Z \asymp Y$ . Further, as  $L' \leq \text{const}$ , Proposition 6.6 gives  $Z' \asymp (L')^{-1}$ , and hence

$$\log(1+Y^{-1}) = \log(1+Y') \asymp Z' \asymp (L')^{-1} \asymp L.$$

Thus,  $\exp[\beta_2 L] \leq 1+Y^{-1} \leq \exp[\beta_1 L]$  for some  $\beta_{1,2} > 0$ , and  $1+Y^{-1} \asymp Y^{-1} \asymp Z^{-1}$ .  $\square$

Now let  $u \in \text{Int } \Omega$  and  $[ab]_\Omega \subset \Omega$  be some boundary arc of  $\Omega$  which should be thought about as lying “very far” from  $u$  (so that the harmonic measure  $\omega_\Omega(u; [ab]_\Omega)$  is small). In order to be able to apply exponential estimate (7.2) to this harmonic measure, one should firstly compare the partition function of random walks running from  $u$  to  $[ab]_\Omega$  in  $\Omega$  with a partition function of random walks running between opposite sides of some quadrilateral. Recall that we denote by  $d_\Omega(u)$  the (Euclidean) distance from  $u$  to  $\partial\Omega$  and  $B_\Omega = B_\Omega(u)$  is a discrete disc around  $u$  of radius  $\frac{1}{4}d_\Omega(u)$ :

$$\text{Int } B_\Omega(u) := \{v \in \text{Int } \Omega : |v-u| < \tfrac{1}{4}d_\Omega(u)\}.$$

Let a discrete domain  $A_\Omega = A_\Omega(u)$  be defined by

$$\text{Int } A_\Omega(u) := \text{Int } \Omega \setminus \text{Int } B_\Omega(u).$$

Note that, according to our conventions, one can identify the inner part of  $\partial A_\Omega$  which we denote by  $C_\Omega = C_\Omega(u)$  (i.e.,  $\partial A_\Omega = C_\Omega \cup \partial\Omega$ ) with  $\partial B_\Omega$ .

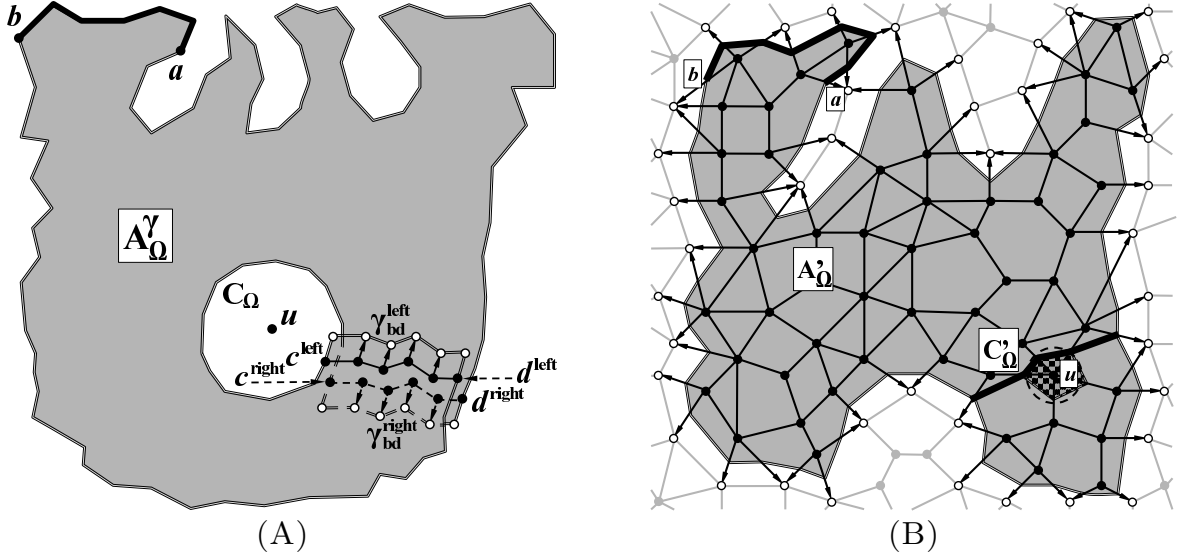


FIGURE 3. (A) In order to analyze the discrete extremal length between  $C_\Omega$  and  $[ab]_\Omega$ , we cut a doubly connected domain  $A_\Omega$  along some nearest-neighbor path  $\gamma$  running from  $c \in C_\Omega$  to  $d \in (ba)_\Omega$ , so that *two* identical copies of  $\gamma$  are included into a simply connected domain  $A_\Omega^\gamma$  (which is drawn on the universal cover of  $A_\Omega$ ). Thus, the boundary  $\partial A_\Omega^\gamma$  is formed by the outer part  $(d^{\text{right}} d^{\text{left}}) = \partial\Omega$ , the inner part  $(c^{\text{left}} c^{\text{right}}) = C_\Omega$  and two paths  $\gamma^{\text{left}}$  and  $\gamma^{\text{right}}$  consisting of vertices neighboring to  $\gamma$ . (B) If a vertex  $u$  is close to  $\partial\Omega$ , it might happen that  $A_\Omega$  is simply connected or even disconnected. Then, we denote by  $A'_\Omega$  the proper connected component of  $A_\Omega$ , and by  $C'_\Omega$  the corresponding part of  $\partial A'_\Omega$ .

**Remark 7.2.** Here and below (Lemma 7.3 – Theorem 7.8) we assume that  $\text{Int } A_\Omega(u)$  is *doubly connected* (in other words,  $B_\Omega(u)$  does not touch the boundary faces of  $\partial\Omega$ , i.e.,  $u$  is not too close to  $\partial\Omega$  in the graph metric). Otherwise, one can apply an appropriate version of Lemma 7.3, which relates  $\omega_\Omega(u; [ab]_\Omega)$  to the partition function of random walks running in  $A_\Omega(u)$ , and directly estimate the latter partition function by the corresponding discrete extremal length using (7.2) (e.g., see the proof of Corollary 7.9).

**Lemma 7.3.** *Let a simply connected discrete domain  $\Omega$  and  $u \in \text{Int } \Omega$  be such that  $A_\Omega(u)$  is doubly connected, and  $[ab]_\Omega \subset \partial\Omega$ . Then,*

$$\omega_\Omega(u; [ab]_\Omega) \asymp Z_{A_\Omega(u)}(C_\Omega(u); [ab]_\Omega). \quad (7.3)$$

*Proof.* For a random walk running from  $u$  to  $[ab]_\Omega$  in  $\Omega$ , let  $v$  denote its last vertex on  $C_\Omega$  (such a vertex exists due to topological reasons). Splitting this path into two halves (before  $v$  and after  $v$ , respectively), one concludes that

$$\omega_\Omega(u; [ab]_\Omega) \asymp Z_\Omega(u; [ab]_\Omega) \asymp \sum_{v \in C_\Omega} Z_\Omega(u; v) Z_{A_\Omega}(v; [ab]_\Omega)$$

As  $Z_\Omega(u; v) = G_\Omega(u; v) \asymp 1$  for any  $v \in C_\Omega$  (see Lemma A.5(ii)), this gives (7.3).  $\square$

In order to relate the partition function (7.3) of random walks in the annulus  $A_\Omega(u)$  to a partition function of random walks in some *simply connected* domain,



below we cut  $A_\Omega(u)$  along appropriate nearest-neighbor paths  $\gamma = (c_{\text{int}} \sim \dots \sim d_{\text{int}})$  such that  $c \in C_\Omega$  and  $d \in \partial\Omega \setminus [ab]_\Omega$ . For a given  $\gamma$  (which is always assumed to be a non-self-intersecting path on the *universal cover*  $A_\Omega^\circ$  of  $A_\Omega$ ), we define a simply connected domain  $\mathbf{A}_\Omega^\gamma$  (see Fig. 3A) as follows:

*if  $\gamma^{\text{left}}, \gamma^{\text{right}}$  are two copies of  $\gamma$  lying on consecutive sheets of  $A_\Omega^\circ$ , then*

$$\text{Int } \mathbf{A}_\Omega^\gamma := \gamma^{\text{left}} \cup [(\text{Int } A_\Omega) \setminus \gamma] \cup \gamma^{\text{right}} \subset \text{Int } A_\Omega^\circ.$$

In other words, we cut  $A_\Omega$  along  $\gamma$ , accounting both sides of the slit as *interior* parts of a discrete domain  $\mathbf{A}_\Omega^\gamma$  (which is, in particular, always connected and simply connected). We then denote by  $\gamma_{\text{bd}}^{\text{left}}$  and  $\gamma_{\text{bd}}^{\text{right}}$  the corresponding parts of  $\partial \mathbf{A}_\Omega^\gamma$ , thus

$$\partial \mathbf{A}_\Omega^\gamma = (d^{\text{left}} d^{\text{right}})_{A_\Omega^\circ} \cup \gamma_{\text{bd}}^{\text{left}} \cup (c^{\text{right}} c^{\text{left}})_{A_\Omega^\circ} \cup \gamma_{\text{bd}}^{\text{right}},$$

where disjoint parts of  $\partial \mathbf{A}_\Omega^\gamma$  are listed counterclockwise with respect to  $\mathbf{A}_\Omega^\gamma$ . We also use simpler notation  $(d^{\text{left}} d^{\text{right}})_{A_\Omega^\circ} = \partial\Omega$  and  $(c^{\text{right}} c^{\text{left}})_{A_\Omega^\circ} = C_\Omega$ , if no confusion arises.

**Corollary 7.4.** *Let a simply connected discrete domain  $\Omega$  and  $u \in \text{Int } \Omega$  be such that  $A_\Omega(u)$  is doubly connected, and  $[ab]_\Omega \subset \partial\Omega$ . Then, for any nearest-neighbor path  $\gamma$  running from  $C_\Omega(u)$  to  $(ba)_\Omega$ , the following is fulfilled:*

$$\text{const} \cdot Z_{\mathbf{A}_\Omega^\gamma}(C_\Omega; [ab]_\Omega) \leq \omega_\Omega(u; [ab]_\Omega) \leq \text{const} \cdot Z_{\mathbf{A}_\Omega^\gamma}(\gamma_{\text{bd}}^{\text{left}} \cup C_\Omega \cup \gamma_{\text{bd}}^{\text{right}}; [ab]_\Omega). \quad (7.4)$$

*Proof.* Indeed,

$$Z_{\mathbf{A}_\Omega^\gamma}(C_\Omega; [ab]_\Omega) \leq Z_{A_\Omega}(C_\Omega; [ab]_\Omega) \leq Z_{\mathbf{A}_\Omega^\gamma}(\gamma_{\text{bd}}^{\text{left}} \cup C_\Omega \cup \gamma_{\text{bd}}^{\text{right}}; [ab]_\Omega)$$

due to simple monotonicity properties of the random walk partition function  $Z_\Omega$  with respect to domain  $\Omega$ : e.g., for the left bound, one forbids the random walks running from  $C_\Omega$  to  $[ab]_\Omega$  to cross  $\gamma$  (still allowing them to touch  $\gamma$  or to run along it).  $\square$

Theorem 7.1 (namely, (7.2)) allows one to estimate both sides of (7.4) using corresponding discrete extremal lengths. We now prove that one can choose  $\gamma$  so that *both* those extremal lengths are comparable to the extremal length of nearest-neighbor paths connecting  $C_\Omega$  and  $[ab]_\Omega$  in the annulus  $A_\Omega$ .

**Remark 7.5.** Below we apply Propositions 6.2 and 6.4 to a doubly connected discrete domain  $A_\Omega$  and its inner boundary  $C_\Omega$  instead of a second boundary arc  $[cd]_\Omega$  of a simply connected domain  $\Omega$ . It is worth to note that we didn't use any "topological" arguments in the proofs of those Propositions.

**Proposition 7.6.** *Let a simply connected discrete domain  $\Omega$  and  $u \in \text{Int } \Omega$  be such that  $A_\Omega(u)$  is doubly connected, and  $[ab]_\Omega \subset \partial\Omega$ . Then,*

(i) *there exists a nearest-neighbor path  $\gamma$  running from  $C_\Omega$  to  $(ba)_\Omega$  such that*

$$L_{\mathbf{A}_\Omega^\gamma}(C_\Omega; [ab]_\Omega) \leq 2L_{A_\Omega}(C_\Omega; [ab]_\Omega);$$

(ii) *for any given  $q > 1$ , either  $L_{A_\Omega}(C_\Omega; [ab]_\Omega) < q^2 L_{A_\Omega}(C_\Omega; \partial\Omega)$  (i.e., the arc  $[ab]_\Omega$  is not so far from  $u$ ), or there exists a nearest-neighbor path  $\gamma$  running from  $C_\Omega$  to  $(ba)_\Omega$  such that*

$$L_{\mathbf{A}_\Omega^\gamma}(\gamma_{\text{bd}}^{\text{left}} \cup C_\Omega \cup \gamma_{\text{bd}}^{\text{right}}; [ab]_\Omega) \geq (1 - q^{-1})^2 L_{A_\Omega}(C_\Omega; [ab]_\Omega).$$

**Remark 7.7.** (i) The constant 2 in the first estimate is an overkill: as it can be seen from the proof, both sides are almost equal to each other for a proper slit  $\gamma$ .

(ii) Since discrete and continuous extremal lengths are uniformly comparable to each other, for any  $\Omega$  and  $u$ , one has

$$L_{A_\Omega}(C_\Omega; \partial\Omega) \asymp L_{A_\Omega^\mathbb{C}}(C_\Omega^\mathbb{C}; \partial\Omega^\mathbb{C}) \asymp 1.$$

*Proof.* Let  $V = V_{(A_\Omega; [ab]_\Omega, C_\Omega)} : A_\Omega \rightarrow [0; 1]$  be the unique discrete harmonic function such that  $V \equiv 0$  on  $[ab]_\Omega$ ,  $V \equiv 1$  on  $C_\Omega$ , and  $V$  satisfies Neumann boundary conditions on  $\partial\Omega \setminus [ab]_\Omega$ . Recall that Proposition 6.4 (see also Remark 7.5) says

$$(L_{A_\Omega}(C_\Omega; [ab]_\Omega))^{-1} = I(V) = \sum_{x \in [ab]_\Omega} w_{xx_{\text{int}}} V(x_{\text{int}}) = \sum_{(vv') \in A_\Omega^\gamma} w_{vv'} (V(v') - V(v))^2.$$

(i) Let  $V^*$  denote a harmonic conjugate function to  $V$  (see (6.8), (6.9) and Remark 6.5) which is defined on the universal cover  $A_\Omega^\circ$  of  $A_\Omega$ . Tracking its increment along  $[ab]_\Omega$ , one easily concludes that  $V^*$  has an additive monodromy  $I(V)$  when passing around  $C_\Omega$  counterclockwise. Moreover, as  $V \in [0; 1]$  everywhere in  $A_\Omega$ , the boundary values of  $V^*$  increases when going counterclockwise along  $C_\Omega$ , as well as along  $\partial\Omega$  (recall that  $V^*$  satisfies Neumann boundary conditions on  $C_\Omega$  and  $[ab]_\Omega$ ).

Let an additive constant in definition of  $V^*$  be chosen so that  $V^* \equiv 0$  on  $\partial\Omega \setminus [ab]_\Omega$  (on some sheet of  $A_\Omega^\circ$ ). Then, there exists a non-self-intersecting nearest-neighbor path  $\gamma$  running from  $C_\Omega$  to  $\partial\Omega \setminus [ab]_\Omega$  in  $A_\Omega^\circ$  which separates non-negative (to the left of  $\gamma$ ) and non-positive (to the right of  $\gamma$ ) values of  $V^*$ . We cut  $A_\Omega$  along  $\gamma$  and choose a branch of  $V^*$  in  $A_\Omega^\gamma$  so that

$$V^* \leq 0 \quad \text{at faces touching } \gamma_{\text{bd}}^{\text{right}}, \quad V^* \geq I(V) \quad \text{at faces touching } \gamma_{\text{bd}}^{\text{left}},$$

$$V^* \equiv 0 \quad \text{at faces touching } [da]_\Omega, \quad V^* \equiv I(V) \quad \text{at faces touching } [bd]_\Omega$$

(recall that  $V^*$  satisfies Neumann boundary conditions on  $[ab]_\Omega$ ). Putting on dual edges  $(ff') = (vv')^*$  of  $A_\Omega^\gamma$  a discrete metric  $g_{ff'} := |V^*(f') - V^*(f)| = w_{vv'} |V(v') - V(v)|$ , one obtains the following estimate for the *dual* discrete length  $L^*$  (see Remark 6.5) between opposite sides  $\gamma^{\text{right}} \cup [da]_\Omega$  and  $[bd]_\Omega \cup \gamma^{\text{left}}$  of  $A_\Omega^\gamma$ :

$$L^* \geq \frac{[I(V)]^2}{\sum_{(vv') \in A_\Omega^\gamma} w_{vv'} |V(v') - V(v)|^2} \geq \frac{[I(V)]^2}{2I(V)} = \frac{1}{2} I(V)$$

(the constant 2 is a big overkill, since each edge of  $A_\Omega$  except  $\gamma$  is counted once in  $A_\Omega^\gamma$ , and only those constituting  $\gamma$  are counted twice). Therefore,

$$L_{A_\Omega^\gamma}(C_\Omega; [ab]_\Omega) = (L^*)^{-1} \leq 2[I(V)]^{-1} = 2L_{A_\Omega}(C_\Omega; [ab]_\Omega).$$

(ii) Let  $d \in \partial\Omega \setminus [ab]_\Omega$  be a boundary vertex where  $V$  attains its maximum on  $\partial\Omega$  (recall that  $V \equiv 0$  on  $[ab]_\Omega$ ). If  $V(d) < 1 - q^{-1}$ , then the metric  $g_{vv'} := |V(v') - V(v)|$  (which is extremal for the family  $(A_\Omega; C_\Omega \leftrightarrow [ab]_\Omega)$ , see Proposition 6.4) provides an estimate

$$L_{A_\Omega}(C_\Omega; \partial\Omega) > \frac{q^{-2}}{I(V)} = q^{-2} L_{A_\Omega}(C_\Omega; [ab]_\Omega).$$

If  $V(d) \geq 1 - q^{-1}$ , let  $\gamma$  denote a nearest-neighbor path running from  $d$  to  $C_\Omega$  such that  $V \geq 1 - q^{-1}$  along this path ( $\gamma$  exists due to the maximum principle). Then, the

same metric as above (we assign zero weights to all edges constituting  $\gamma^{\text{left}}, \gamma^{\text{right}}$  and corresponding boundary ones) gives

$$L_{A_\Omega}(\gamma_{\text{bd}}^{\text{left}} \cup C_\Omega \cup \gamma_{\text{bd}}^{\text{right}}; [ab]_\Omega) \geq \frac{(1-q^{-1})^2}{I(V)} = (1-q^{-1})^2 L_{A_\Omega}(C_\Omega; [ab]_\Omega). \quad \square$$

Combining estimates given above, we are now able to prove a uniform double-sided estimate relating the logarithm of discrete harmonic measure  $\omega_\Omega(u; [ab]_\Omega)$  in a simply connected  $\Omega$  and discrete extremal length  $L_{A_\Omega}(C_\Omega; [ab]_\Omega)$  in the annulus  $A_\Omega(u)$ .

**Theorem 7.8.** *Let a simply connected discrete domain  $\Omega$  and  $u \in \text{Int } \Omega$  be such that  $A_\Omega(u)$  is doubly connected, and  $[ab]_\Omega \subset \partial\Omega$ . Then,*

$$\log(1 + (\omega_\Omega(u; [ab]_\Omega))^{-1}) \asymp L_{A_\Omega(u)}(C_\Omega(u); [ab]_\Omega), \quad (7.5)$$

with constants independent of  $\Omega$ ,  $u$  and  $[ab]_\Omega$ .

*Proof.* Let  $L := L_{A_\Omega}(C_\Omega; [ab]_\Omega)$  and  $\omega := \omega_\Omega(u; [ab]_\Omega)$ . Corollary 7.4, Theorem 7.1 and Proposition 7.6 provide us the following diagram (for some proper discrete cross-cuts  $\gamma$  which can be different for lower and upper bounds):

$$\begin{array}{ccc} \text{const} \cdot Z_{A_\Omega}^\gamma(C_\Omega; [ab]_\Omega) & \leq \omega \leq & Z_{A_\Omega}^\gamma(\gamma_{\text{bd}}^{\text{left}} \cup C_\Omega \cup \gamma_{\text{bd}}^{\text{right}}; [ab]_\Omega) \\ \updownarrow & & \updownarrow \\ 2L \geq L_{A_\Omega}^\gamma(C_\Omega; [ab]_\Omega) & & L_{A_\Omega}^\gamma(\gamma_{\text{bd}}^{\text{left}} \cup C_\Omega \cup \gamma_{\text{bd}}^{\text{right}}; [ab]_\Omega) \geq \frac{1}{2} L \end{array}$$

(the last inequality holds true, if  $L \geq \lambda_0$ , where  $\lambda_0$  is some absolute constant: recall that  $L_{A_\Omega}(C_\Omega; \partial\Omega) \asymp 1$  for all  $\Omega$  and  $u$ ). Above, “ $\updownarrow$ ” means a double-sided estimate of  $Z_{A_\Omega}^\gamma$  via  $L_{A_\Omega}^\gamma$  given by Theorem 7.1 (note that it is *inverse* monotone: an upper bound for  $L_{A_\Omega}^\gamma$  gives a lower bound for  $Z_{A_\Omega}^\gamma$  and vice versa).

In particular, if  $L \geq \lambda_0$ , condition (7.1) holds for both (right, and therefore, left) columns. Thus, in this case, one can replace both “ $\updownarrow$ ” by (7.2), arriving at  $\log \omega \asymp -L$ . If  $L < \lambda_0$ , then the left column gives  $\omega \geq \text{const}$  and both sides of (7.5) are uniformly comparable to 1 (note that  $L$  is uniformly bounded below by  $L_{A_\Omega}(C_\Omega; \partial\Omega) \asymp 1$ ).  $\square$

**Corollary 7.9.** *Let  $\Omega$  be a simply connected domain,  $u \in \text{Int } \Omega$ , and  $[ab]_\Omega \in \partial\Omega$ . Denote  $\omega_{\text{disc}} := \omega_\Omega(u; [ab]_\Omega)$  and  $\omega_{\text{cont}} := \omega_\Omega^\mathbb{C}(u; [ab]_\Omega^\mathbb{C})$ . Then,*

$$\log(1 + \omega_{\text{disc}}^{-1}) \asymp \log(1 + \omega_{\text{cont}}^{-1})$$

with some absolute (i.e., independent of  $\Omega, u, a, b$ ) constants.

*Proof.* First, let us assume that  $A_\Omega(u)$  is doubly connected, so  $\Omega$  and  $u$  fit the setup of Theorem 7.8. Let  $L_{\text{disc}} := L_{A_\Omega}(C_\Omega; [ab]_\Omega)$  and  $L_{\text{cont}} := L_{A_\Omega^\mathbb{C}}(C_\Omega^\mathbb{C}; [ab]_\Omega^\mathbb{C})$  be its continuous counterpart. Recall that  $L_{\text{disc}} \asymp L_{\text{cont}}$  due to Proposition 6.2 (and Remark 7.5). Then,

$$\log(1 + \omega_{\text{disc}}^{-1}) \asymp L_{\text{disc}} \asymp L_{\text{cont}} \asymp \log(1 + \omega_{\text{cont}}^{-1}),$$

where the last estimate is an easy corollary of the classical estimates for harmonic measure via extremal lengths (e.g., see [GM05, Theorem 5.2]).

If  $A_\Omega(u)$  is *not* doubly connected, then at least one of vertices  $v \in \partial B_\Omega(u)$  belongs to a face touching  $\partial\Omega$ . In particular, this gives  $r_{v_{\text{int}}} \asymp d_\Omega(v_{\text{int}}) \asymp d_\Omega(u)$ . Thus, the number of vertices  $\#B_\Omega(u)$  is uniformly bounded (see Remark 2.2(ii)). Further, if at least one of vertices of  $\partial B_\Omega(u)$  shares a face with  $[ab]_\Omega$ , then  $\omega_{\text{disc}} \geq \text{const}$  and

$\omega_{\text{cont}} \geq \text{const}$  as well (indeed, the Brownian motion started at  $u$  has a positive chance to hit  $[ab]_{\mathbb{C}}$  traveling nearby of the corresponding finite length lattice path).

Thus, without loss of generality, we may assume that both  $\omega_{\text{disc}}$  and  $\omega_{\text{cont}}$  are uniformly bounded away of 1 and there exists a connected (and simply connected) component of  $\text{Int } A_{\Omega}(u)$  whose boundary contains the whole arc  $[ab]_{\Omega}$ . Let  $A'_{\Omega}$  denote this component of  $A_{\Omega}$  and  $C'_{\Omega} \subset \partial A'_{\Omega}$  be the corresponding part of  $C_{\Omega}$  slightly enlarged so that it includes two nearby boundary points of  $\partial\Omega$  (see Fig. 3B). Further, let  $L'_{\text{disc}} := L_{A'_{\Omega}}(C'_{\Omega}; [ab]_{\Omega})$  and  $L'_{\text{cont}} := L_{A'_{\Omega}^c}(C'_{\Omega}^c; [ab]_{\Omega}^c)$  denote its continuous counterpart. It is easy to see that one still has

$$\omega_{\text{disc}} \asymp Z_{\Omega}(u; [ab]_{\Omega}) \asymp Z_{A'_{\Omega}}(C_{\Omega} \cap \partial A'_{\Omega}; [ab]_{\Omega}) \asymp Z_{A'_{\Omega}}(C'_{\Omega}; [ab]_{\Omega})$$

(the proof of Lemma 7.3 works well without any changes, and replacing  $C_{\Omega} \cap \partial A'_{\Omega}$  by  $C'_{\Omega}$  costs no more than an absolute multiplicative constant). Applying (7.2) and Proposition 6.2, one obtains

$$\log \omega_{\text{disc}} \asymp -L'_{\text{disc}} \asymp -L'_{\text{cont}} \asymp \log \omega_{\text{cont}},$$

(to prove the last estimate, e.g., draw a circle  $c_u \subset \Omega^c$  of radius  $\frac{1}{4}r_u \asymp d_{\Omega}(u)$  around  $u$ , then  $-\log \omega_{\text{cont}} \asymp L_{\Omega^c}(c_u; [ab]_{\Omega}^c) \asymp L'_{\text{cont}}$ ).  $\square$

## A. APPENDIX

In this Appendix we show how to derive several statements of discrete potential theory used in the main text from two Assumptions (S) and (T) made in Section 2.3.

**A.1. Assumption (S), elliptic Harnack principle, and weak Beurling estimate.** We start with a simple Lemma A.1 which is an immediate corollary of (S), and then use it to derive (uniform) elliptic Harnack and weak Beurling estimates.

**Lemma A.1.** *There exist two constants  $\delta_0, c_0 > 0$  such that the following holds true: if  $u \in \Gamma$ ,  $r \geq r_u$  are such that  $r_v \leq \delta_0 r$  for all  $v \in B_r^{\Gamma}(u)$  (cf. Remark 2.2), and  $w \in \Gamma : \frac{2}{3}r \leq |w - u| \leq \frac{5}{6}r$ , then the probability of the event that the random walk started at  $w$  makes a whole turn inside of the annulus  $B_r^{\Gamma}(u) \setminus B_{r/2}^{\Gamma}(u)$  (and cross its own trajectory afterwards) is uniformly bounded below by  $c_0 > 0$ .*

*Proof.* Indeed, there exists a small constant  $\tau_0 > 0$  (depending on constants in assumptions (a)–(c) in Sect. 2.1 only) such that applying (S) to a uniformly bounded number discs of radius  $\tau_0 r$  and appropriate boundary arcs of angle  $\pi - \eta_0$  step by step, one can “drive” the random walk started at  $w$  so that it makes the full turn staying inside of  $B_r^{\Gamma}(u) \setminus B_{r/2}^{\Gamma}(u)$  (cf. [CS11, Proof of Proposition 2.11]).  $\square$

**Lemma A.2 (elliptic Harnack principle).** *Let  $H : \Omega \rightarrow [0, +\infty)$  be a nonnegative discrete harmonic function defined in  $\Omega$ . Then, for any  $u \in \text{Int } \Omega$ ,*

$$H(v) \asymp H(v') \quad \text{uniformly for all } v, v' \in \text{Int } B_{d_{\Omega}(u)/2}^{\Gamma}(u),$$

where  $d_{\Omega}(u)$  denotes the (Euclidean) distance from  $u$  to  $\partial\Omega$ .

*Proof.* Note that, without loss of generality, we can assume that  $r_v \leq \delta_0 d_{\Omega}(u)$  for all  $v \in B_r^{\Gamma}(u)$  (see Remark 2.2, otherwise the number of vertices in  $B_{d_{\Omega}(u)}^{\Gamma}(u)$  is uniformly bounded, hence all values of  $H$  in this disc are uniformly comparable to each other, as all vertices can be joined by a uniformly bounded number of edges). Further, by

the maximum principle, there exists a nearest-neighbor path  $\gamma_v$  running from  $v$  to  $\partial\Omega$  such that  $H \geq H(v)$  along this path. Then, Lemma A.1 implies  $H(w) \geq c_0 H(u)$  for all  $w : \frac{2}{3}d_\Omega(u) \leq |w-u| \leq \frac{5}{6}d_\Omega(u)$  (as the random walk traveling around the annulus should cross  $\gamma_v$  at some point). Then, the maximum principle gives  $H(v') \geq c_0 H(v)$  for all  $v' \in \text{Int } B_{d_\Omega(u)/2}^\Gamma(u)$ . The inverse estimate is similar.  $\square$

**Remark A.3.** Using the same argument, one can prove the following one-sided version of Harnack's estimate *near the boundary* of a discrete domain  $\Omega$  which was used in the proof of Lemma 3.2 (recall that  $B_r^\Omega(x)$  denotes the  $r$ -neighborhood of  $x$  in  $\Omega$ ):

Let  $x \in \partial\Omega$  and  $r \geq r_x$  be such that  $r_v \leq \delta_0 r$  for all  $v \in B_r^\Gamma(x)$ , and  $w \in B_r^\Omega(x)$  be such that  $\frac{2}{3}r \leq |w-u| \leq \frac{5}{6}r$ . If  $H$  is a nonnegative discrete harmonic function in  $B_r^\Omega(x)$  which *vanishes on*  $\partial\Omega \cap \partial B_r^\Omega(x)$ , then  $H(v) \leq c_0^{-1} H(w)$  for all  $v \in \text{Int } B_{r/2}^\Omega(x)$ .

Indeed, due to the maximum principle, one has  $H \geq H(v)$  along some nearest-neighbor path  $\gamma_v$  running from  $v$  to the outer boundary of  $B_r^\Omega(x)$  (this path cannot end on  $\partial\Omega \cap \partial B_r^\Omega(x)$  as  $H$  vanishes there). Thus, Lemma A.1 implies  $H(w) \geq c_0 H(v)$  since the random walk started at  $w$  has a positive chance to hit  $\gamma_v$  before  $\partial\Omega$ .

**Lemma A.4 (weak Beurling estimate).** *There exists an absolute constant  $\beta_0 > 0$  such that, for any simply connected discrete domain  $\Omega$ ,  $u \in \text{Int } \Omega$  and  $E \subset \partial\Omega$ , the following is fulfilled:*

$$\omega_\Omega(u; E) \leq \text{const} \cdot \left[ \frac{d_\Omega(u)}{\text{dist}_\Omega(u; E)} \right]^{\beta_0} \quad \text{and} \quad \omega_\Omega(u; E) \leq \text{const} \cdot \left[ \frac{\text{diam } E}{\text{dist}_\Omega(u; E)} \right]^{\beta_0},$$

where

$$\text{dist}_\Omega(u; E) := \inf\{r : u \text{ and } E \text{ are connected in } \Omega \cap B_r^\Gamma(u)\}.$$

Above we set  $\text{diam } E := r_x$ , if  $E = \{x\}$  consists of a single vertex.

*Proof.* The proof mimics the proof of Proposition 2.11 in [CS11]. Using Lemma A.1, it is easy to conclude that the random walk started at  $u$  has a chance  $c_0 > 0$  to hit  $\partial\Omega \setminus E$  before crossing the annulus  $B_{2^k d_\Omega(u)}^\Gamma \setminus B_{2^{k-1} d_\Omega(u)}^\Gamma$  for any  $k \leq \log_2[\text{dist}_\Omega(u; E)/d_\Omega(u)]$  (if some annulus contains a vertex which local scale size is comparable to  $2^k d_\Omega(u)$ , then there are only a uniformly bounded number of vertices in  $B_{2^k d_\Omega(u)}^\Gamma$ , and so the random walk can hit  $\partial\Omega \setminus E$  by a uniformly bounded number of steps). This gives the first estimate with  $\beta_0 := -\log_2(1-c_0)$ , and the second follows from the same arguments (applied to the random walk started on  $E$  and running to  $u$ ).  $\square$

**A.2. Assumption (T) and the pointwise estimate for Green's function.** For a simply connected domain  $\Omega$ ,  $u \in \text{Int } \Gamma$  and  $r > 0$ , denote

$$B_\Omega^{(r)}(u) := B_{rd_\Omega(u)}^\Gamma(u) \quad \text{and} \quad C_\Omega^{(r)}(u) := \{x_{\text{int}} \in \Gamma : (x; (x_{\text{int}}x)) \in \partial B_\Omega^{(r)}\}.$$

Also, let

$$G_\Omega^{(r)}(v; u) := G_{B_\Omega^{(r)}}(v; u)$$

denotes Green's functions in the discrete disc of radius  $rd_\Omega(u)$  around  $u$ .

**Lemma A.5.** *Let  $\Omega$  be a simply connected discrete domain and  $u \in \text{Int } \Omega$ . Then,*  
*(i) for any  $v \in \text{Int } B_\Omega^{(1)}(u)$ , one has*

$$G_\Omega^{(1)}(v; u) \leq G_\Omega(v; u) \leq c_0^{-1} G_\Omega^{(2)}(v; u);$$

*(ii) for any given  $0 < r < 1$ , there exist two constants  $c(r), C(r) > 0$  (independent of  $\Omega$  and  $u$ ) such that the following holds true:*

$$c(r) \leq G_\Omega(v; u) \leq C(r) \quad \text{for all } v \in C_\Omega^{(r)}(u).$$

*Proof.* Note that, without loss of generality, we can assume that  $r_v \leq \delta d_\Omega(u)$  for all  $v \in B_\Omega^{(2)}(u)$ , where  $\delta$  is chosen small enough (otherwise, there is only a uniformly bounded number of points in  $B_\Omega^{(2)}(u)$ , so all values of Green's functions  $G_\Omega^{(1)}, G_\Omega, G_\Omega^{(2)}$  are uniformly comparable with those at  $v = u$ , and  $G_\Omega^{(1)}(u; u) \asymp G_\Omega(u; u) \asymp G_\Omega^{(2)}(u; u) \asymp 1$  since the random walk started at  $u$  has a positive chance to hit boundary by a uniformly bounded number of steps before coming back to  $u$  next time).

(i) The first estimate  $G_\Omega^{(1)} \leq G_\Omega$  is trivial as Green's function is monotone with respect to  $\Omega$ , so we need to prove that  $G_\Omega \leq c_0^{-1} G_\Omega^{(2)}$ . Let  $v \in C_\Omega^{(r)}(u)$  for some  $r \leq 1$ . Using Lemma A.1, it is easy to conclude that

$$G_\Omega(v; u) \leq G_\Omega^{(2)}(v; u) + (1 - c_0) \max_{v \in C_\Omega^{(r)}(u)} G_\Omega(v; u).$$

Indeed, if the random walk started at  $v$  reaches  $\partial B_\Omega^{(2)}(u)$  before hitting  $\partial\Omega$ , then it has a positive chance  $c_0 > 0$  to hit  $\partial\Omega$  before coming back to  $C_\Omega^{(r)}(u)$  (in order to have a possibility to visit  $u$  and to contribute to  $G_\Omega(v; u)$  again). Thus, for any  $r \leq 1$ ,

$$\max_{v \in C_\Omega^{(r)}(u)} G_\Omega(v; u) \leq c_0^{-1} \max_{v \in C_\Omega^{(r)}(u)} G_\Omega^{(2)}(v; u).$$

(ii) To prove the upper bound in (A.1), note that, for any  $v \in C_\Omega^{(r)}(u)$ , one has  $G_\Omega(\cdot; u) \geq G_\Omega(v; u)$  along some nearest-neighbor path running from  $v$  to  $u$ . As above, applying Lemma A.1 and the maximum principle, we get

$$G_\Omega(w; u) \geq c_0 G_\Omega(v; u) \quad \text{uniformly for } w \in \text{Int } B_\Omega^{5r/6}(u), \quad (\text{A.1})$$

which implies  $G_\Omega(v; u) \leq \text{const}$  due to  $G_\Omega^{(2)}(w; u) \geq c_0 G_\Omega(w; u)$  and the uniform upper bound in Assumption (T).

To prove the lower bound in (A.1), note that  $m_r := \max_{v \in C_\Omega^{(r)}(u)} G_\Omega^{(1)}(v; u)$  is uniformly bounded below, if  $r$  is small enough. Indeed, the maximum principle gives

$$\begin{aligned} G_\Omega^{(1)}(v; u) &\leq m_r && \text{outside of } C_\Omega^{(r)}(u), \\ G_\Omega^{(1)}(v; u) &\leq m_r + G_\Omega^{(r)}(v; u) && \text{inside } C_\Omega^{(r)}(u). \end{aligned}$$

Hence, Assumption (T) applied for both discs  $B_\Omega^{(1)}(u)$  and  $B_\Omega^{(r)}(u)$  implies

$$\text{const} \cdot (d_\Omega(u))^2 \leq \text{const} \cdot m_r^2 (d_\Omega(v))^2 + \text{const} \cdot (rd_\Omega(v))^2.$$

Therefore, there exist a constant  $r_0 > 0$  (independent of  $\Omega$  and  $u$ ) such that  $G_\Omega^{(1)}(v; u) \geq \text{const} > 0$  at least at one point  $v \in C_\Omega^{(r_0)}(u)$ . Applying Harnack's principle (Lemma A.2) several times (depending on  $r_0$  and  $r$  but not on  $\Omega$  and  $u$ ), one gets the uniform lower bound for  $G_\Omega(\cdot; u) \geq G_\Omega^{(1)}(\cdot; u)$  on  $C_\Omega^{(r)}(u)$  for any given  $0 < r < 1$ .  $\square$

**Remark A.6.** In the proof of Proposition 3.1, we used the following uniform estimate:

$$\sum_{v \in \text{Int } B_\Omega(u)} r_v^2 G_\Omega(v; w) \asymp (d_\Omega(u))^2 \quad \text{for any } w \in \text{Int } B_\Omega(u).$$

The lower bound is an easy corollary of the lower bound in (A.1). The upper bound follows from (i) and the upper bound in (A.1) applied for  $G_\Omega^{(2)}(v; w)$ .

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